DOCUMENT RESUME

ED 094 995 SE 017 535

AUTHOR	Luce, Marjory; Muckey, Roy
TITLE	Mathematics for the Elementary School, Unit 17, Addition and Subtraction in Squareville.
INSTITUTION	Minnesota Univ., Minneapolis. Minnesota School Mathematics and Science Center.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	65
NOTE	65p.
EDRS PRICE	MF-$0.75 HC-$3.15 PLUS POSTAGE
DESCRIPTORS	Activity Learning; *Addition; Curriculum; *Elementary School Mathematics; Graphs; Instruction; *Instructional Materials; Number Concepts; *Subtraction; Symmetry; *Teaching Guides; Units of Study (Subject Fields); Worksheets
IDENTIFIERS	MINNEMAST; *Minnesota Mathematics and Science Teaching Project

ABSTRACT

The Minnesota School Mathematics and Science Teaching (MINNEMAST) Project is characterized by its emphasis on the coordination of mathematics and science in the elementary school curriculum. Units are planned to provide children with activities in which they learn various concepts from both subject areas. Each subject is used to support and reinforce the other where appropriate, with common techniques and concepts being sought and exploited. Content is presented in story fashion. The stories serve to introduce concepts and lead to activities. Imbedded in the pictures that accompany the stories are examples of the concepts presented. This unit provides students with a graphic method of addition which is useful both as a checking device and as an advance in what the student did in Unit 16. Three different techniques of using graphs and linear functions for addition are presented. This approach integrates ideas of geometric symmetry and with number relationships under addition and subtraction. Worksheets and commentaries to the teacher are provided and additional activities are suggested. (JP)

MINNE
Math And Science Teaching project

UNIT XVII
ADDITION & SUBTRACTION IN SQUAREVILLE

MINNESOTA SCHOOL MATHEMATICS AND SCIENCE CENTER 1965

MATHEMATICS

FOR THE

ELEMENTARY SCHOOL

UNIT XVII

Addition and Subtraction in Squareville

The Minnesota School Mathematics and Science Teaching Project

provided these materials under a grant from the

National Science Foundation

second printing
1965

MINNESOTA MATHEMATICS AND SCIENCE TEACHING PROJECT

JAMES H. WERNTZ, JR.
Associate Professor of Physics
University of Minnesota
Project Director

PAUL C. ROSENBLOOM
Professor of Mathematics
Teachers College, Columbia University
Mathematics Director

Addition and Subtraction in Squareville

MARJORY LUCE — WRITER
Elementary School Teacher
Edina Schools, Minnesota

ROY MUCKEY — WRITER
Minnemast Staff
Elementary School Teacher
Roseville Schools, Minnesota

ELAINE PARENT — CONTEXT EDITOR
Minnemast Staff
University of Minnesota

CLAUDE SCHOCHET — MATH CONTENT EDITOR
Minnemast Staff
University of Minnesota

MARY LOU KNIPE — ASSISTANT
Minnemast Staff
University of Minnesota

RUTH EDER — ASSISTANT
Minnemast Staff
University of Minnesota

We are deeply indebted to the many teachers who
used earlier versions of this material
and provided suggestions for
this revision

CONTENTS

INTRODUCTION

Teacher Background

By the end of Unit 15, Addition and Linear Translations, the student has learned to add two-digit positive integers in two ways:

a. geometrically - by using blocks or intervals and a slide rule

b. algebraically - in the form of $a + b = c$.

In Unit 16, The Story of Squareville, the student has become acquainted with the ordered pair as the name of the intersection of two perpendicular lines in a plane. Highways $y = x$ and $y = x + 1$ are named.

Unit 17, Addition and Subtraction in Squareville, provides the student with a graphic method of addition which is useful both as a checking device and as an advance in what the student can do in Squareville.

The student is introduced to three techniques for using Squareville as an adding machine. Tommy's method locates the answer on the "y axis". Ellen's method locates the answer on the "x axis". Ethelbert's method, a generalization of Ellen's, is the most useful for visualizing the addition properties and for adding several numbers together. His answers are also located on the x axis.

As the student adds and subtracts in Squareville, he finds it necessary to build many new addition and subtraction highways. In Unit 17, the student is seldom forced to go to the left of the y axis or below the x axis and, thus, encounter coordinates involving negative numbers, although it might be appropriate for the teacher to raise questions along these lines. Negative numbers will be officially introduced in a later unit. It is hoped that students will already understand this concept, having had frequent experiences with it (especially in Unit XXI).

All the highways that are built in this unit are straight lines parallel to the $y = x$ line. By letting $y = x$ be the axis of symmetry, the student discovers relations of symmetry among the highways that are used for addition and subtraction.

The mathematical experience of the classroom teacher will undoubtedly determine the extent to which she associates graphing in Squareville with the idea of a "function". A function is a rule by which every member of a set A is associated with a unique member of another set B. For example, letting A = real numbers and B = real numbers, the function $f(x) = 3x$ associates 1 with 3, 4 with 12, 0 with 0, etc.

The story in the unit is interspersed with teacher comments, suggested activities, and worksheets. The reasons for this are as follows:

1. The teacher is to read or tell the story to the students. As she does, they will be introduced to the same problem as the characters in the story. Teacher's comments are given which might suggest some procedures for encouraging the class to search for their own solutions to the problem.
2. It is hoped that the teacher will solicit an extensive class discussion during which alternative solutions are suggested and evaluated.
3. After the class has discussed the problem, the story continues and a solution is introduced. The students should then compare their solutions with the one in the story. They must be allowed to discuss any disagreements they might have with the story.
4. Then the story goes on to develop another problem. The children are allowed to solve that problem and then compare their solutions with the one in the story.
5. The teacher, after finishing the story, might let the students review the ideas by reading the story at their seats.

6. The story is also intended to serve as a means for informing the parents of the concepts encountered by the child. Eventually these stories will be given to the children to take home.

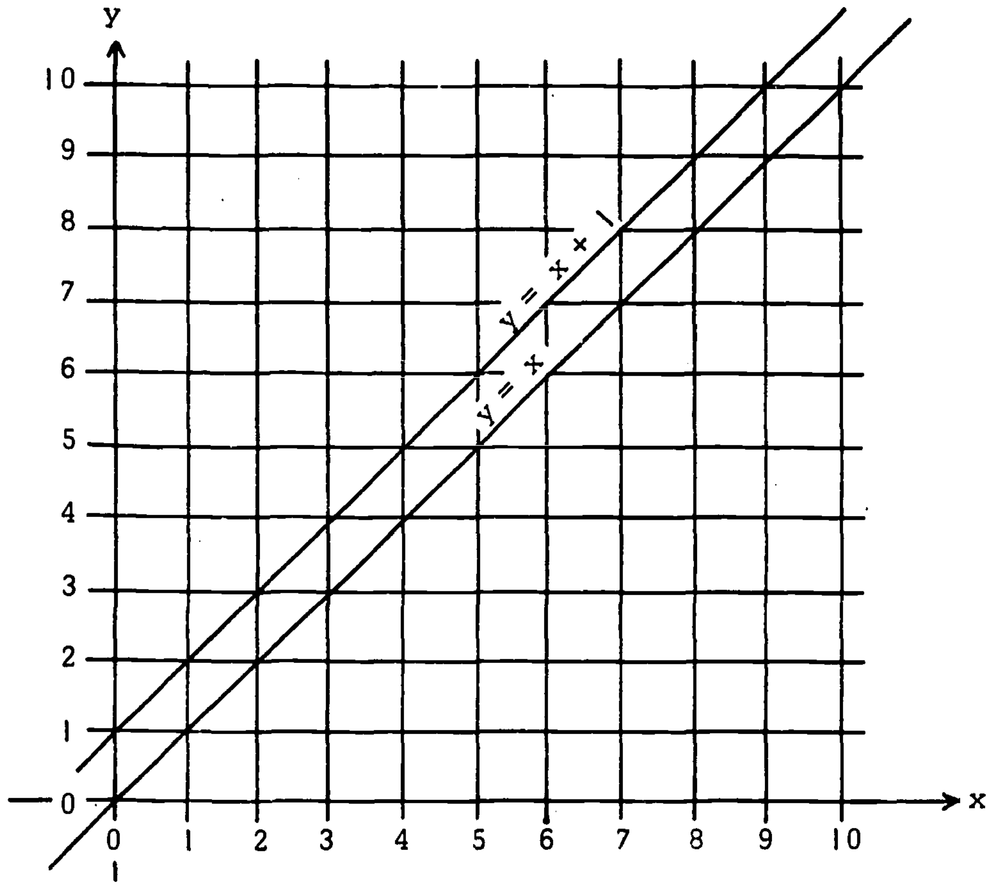
The success of the Minnemath material depends on getting each child personally involved in the discovery of new ideas to the greatest possible extent. His ideas, accurate or not, must be respected. Then he will dare to state his own opinions so that they might be reasonably evaluated. Through the use of a variety of techniques such as "whisper your answer to the teacher", "show it to the class", "work with a partner", and others, each child can have a maximum number of tries at solving each problem.

TOMMY ADDS IN SQUAREVILLE

That same night the Mayor showed his plans for the $y = x + 1$ highway to his son, Tommy, who looked at them for a moment. Suddenly Tommy said, "Gee! I could do my arithmetic on your map. How about lending it to me so that I can show the kids at school?"

Of course the Mayor didn't believe him, but he gave Tommy the map anyway and said, "I'd advise you not to go around telling fairy tales."

Tommy took the map and looked at it.



Then he looked at his dad and said, "But I'm not fibbing. If I want to add 1 to any number, all I need to do is use Highway $y = x + 1$."

Ask the children if they can suggest ways of using y = x + 1 to add 1 to any number. Be sure to discuss their suggestions. Then continue with the story. As the story continues and the map is discussed, have some child mark the points and follow the action.

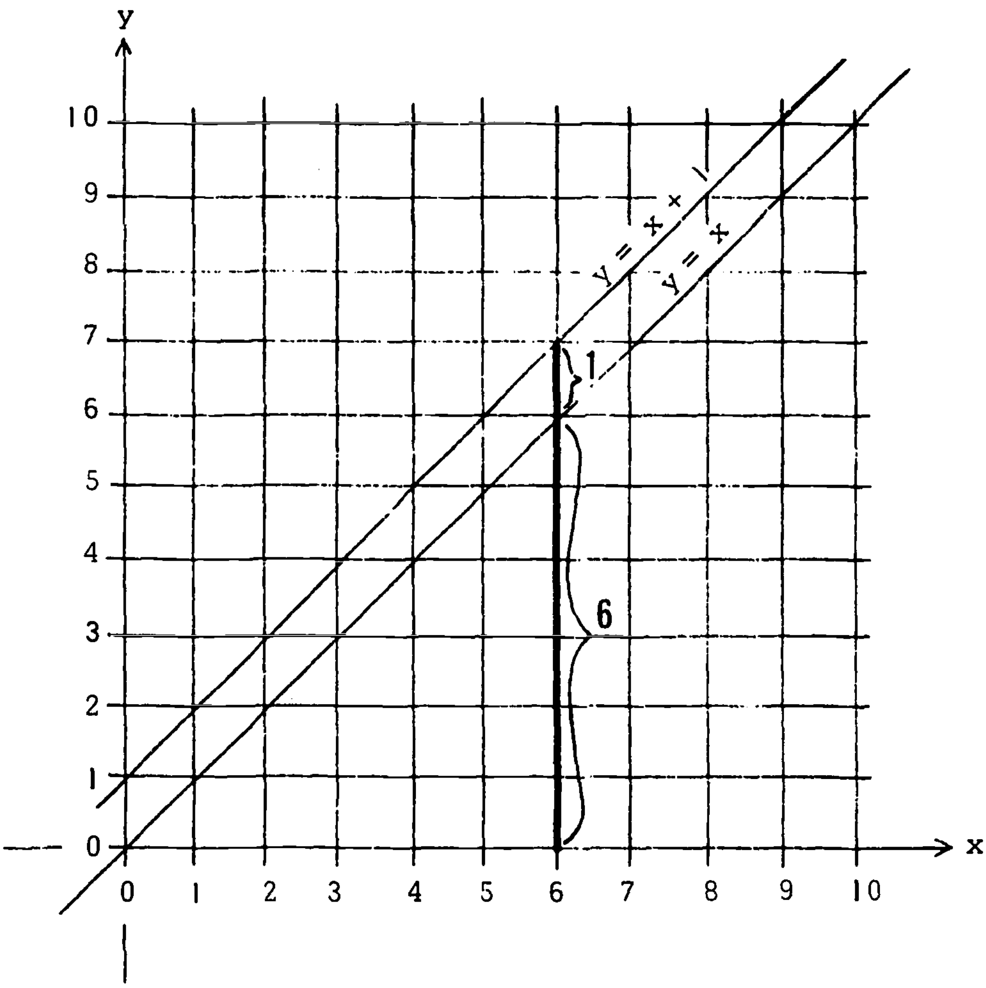
Tommy continued, "I said that I can add 1 to any number by using the y = x + 1 highway. Suppose I decide to add 1 to 6. I will begin at (6, 0).

From (6, 0), north, to y = x is 6 blocks.

Then from y = x, north, to y = x + 1 is 1 block.

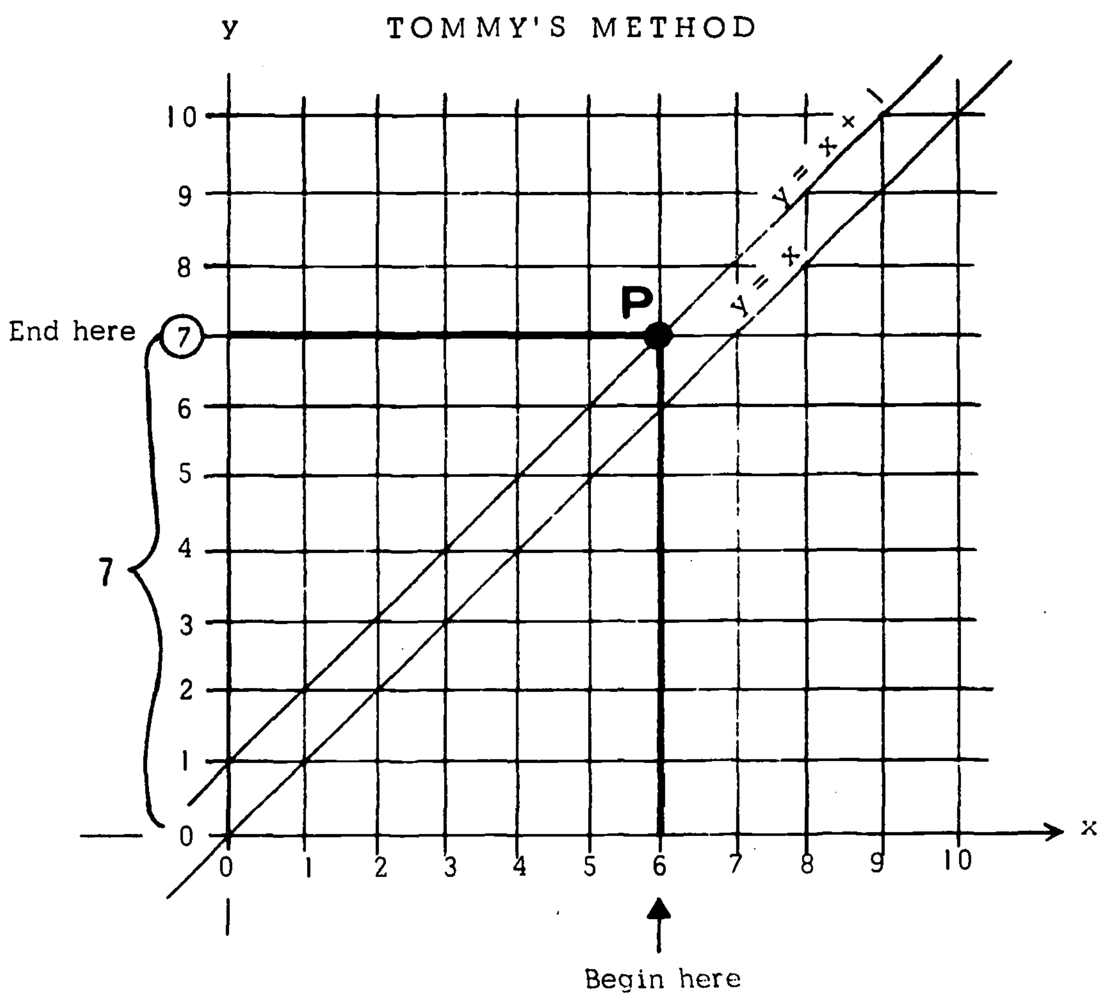
So from (6, 0), north, to y = x + 1 is 6 + 1 blocks.

It looks like this on Squareville," and he drew the accompanying map.



"I get it," said the Mayor. "You use the y = x + 1 highway when you add 1 to any number. But where do you read your answer?"

"Easy," Tommy replied. "6 + 1 places me at a certain point on 6th Street. Let's call that point "P". The sum 6 + 1 is the avenue address of <u>point P</u>. If I look to the left, I see that P is on 7th Avenue. So the sum of 6 and 1 is 7."



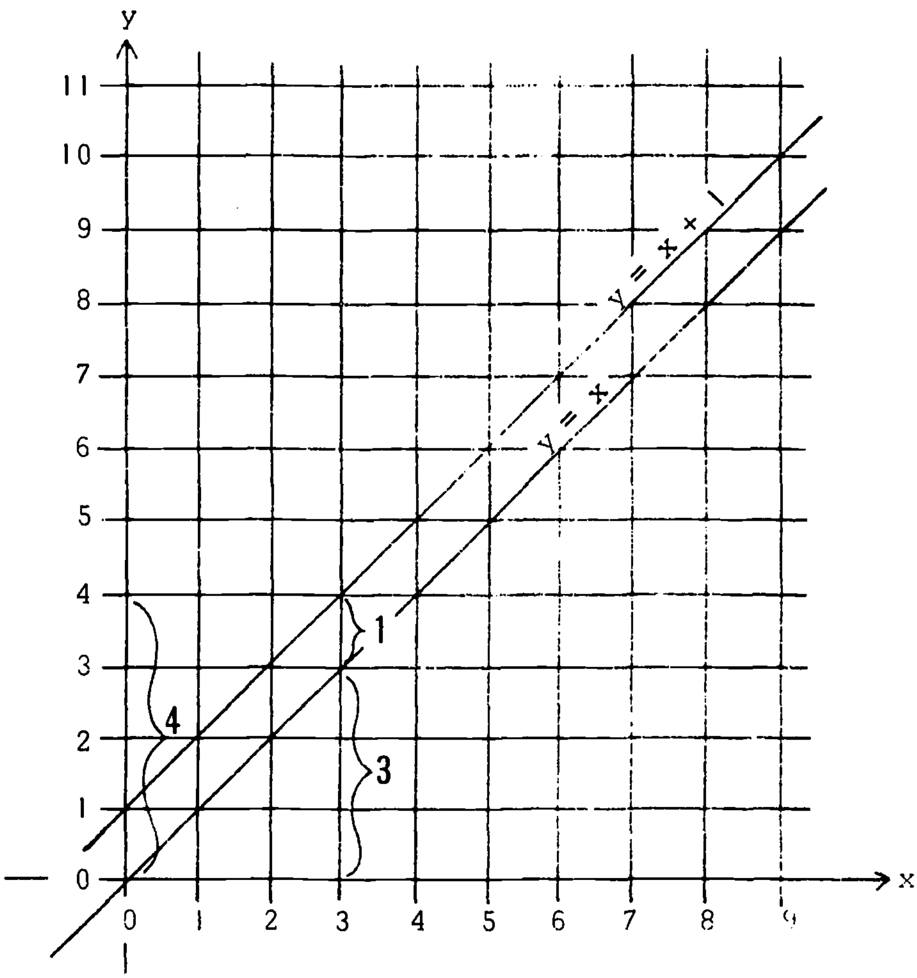
Mr. Brown smiled, "You weren't fibbing. The y = x + 1 highway is very important when you want to add 1 to a number."

Activity 1

1. Provide each child with several maps of Squareville. Have them draw in highways y = x and y = x + 1.

2. Begin by demonstrating how Tommy used the y = x + 1 highway to add 1 to any number. Use different colors of chalk to mark the various line segments.

3. After the chalkboard demonstration, have each child use colors to mark off other examples of adding 1 to any number.



4. Have the children check each others' papers.

TOMMY TELLS HIS CLASSMATES

> From now on 0 Avenue will also be referred to as the "x axis" and 0 Street as the "y axis". You might discuss the reasons for this convenience with the children.

The next morning Tommy hurried to school to show his discovery to his teacher, Mrs. Jones. She liked his method so much that she gave him permission to show it to the class.

When math time came, Tommy proudly told the other boys and girls how he added 1 to any number by using highway $\boxed{y = x + 1}$. He carefully explained how he found the avenue address by looking to the left.

One of the boys, Jimmy, piped up and said, "Back in the first grade we used to use the number line when we added one to any number. This is far more interesting. I like it!"

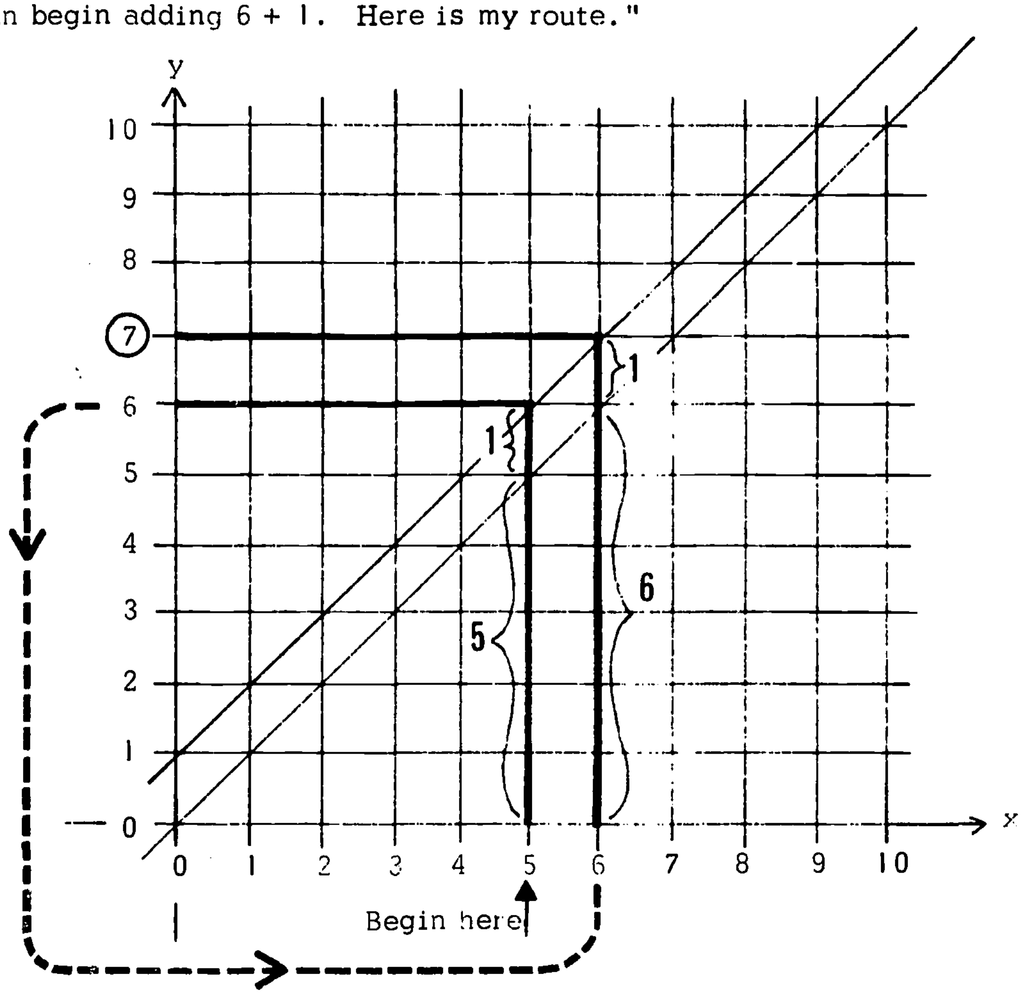
Many of the children wanted to try Tommy's method, so Mrs. Jones gave them some maps of Squareville to use.

After the class had practiced for awhile, Mrs. Jones stopped them and said, "Suppose you start with any number, add 1 to it, and then add 1 again to your answer. Can you do it on Squareville?"

Tommy waved his hand and yelled, "I can do it! Let me show everyone."

> Before reading on, ask if anyone knows how Tommy can do it. Let the students try to illustrate their methods on their own maps and show them to you. After everyone has thought for awhile, let some children demonstrate their techniques on the chalkboard map.

Tommy rushed to the chalkboard map of Squareville and began explaining, "If I compute 5 + 1 + 1 (that is, add 1 to 5, and then add 1 to the first sum), I must find the sum of adding 1 to 5 on the y-axis. Then I go to (6,0) so that I can begin adding 6 + 1. Here is my route."



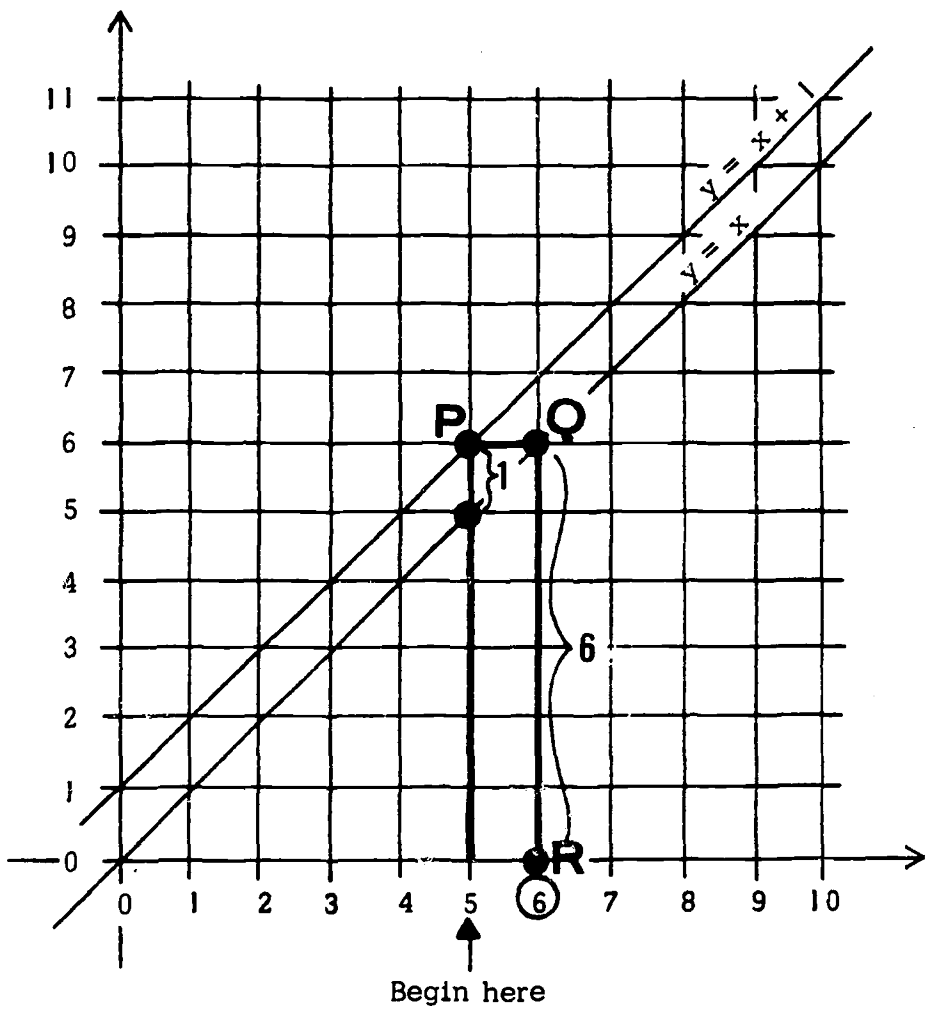
Have each child solve a few of this type of problem. Some children may want to work together. Samples: 6 + 1 + 1 = _______ 4 + 1 + 1 = _______

When Tommy had finished, Ellen spoke to the class, "I have another way of doing it. My way will put you in a better position to do the second addition. Let me show you how I would find the answer for 5 + 1 + 1.

I begin at (5, 0), which is on the x-axis. Tommy also begins at this point.

Then I go north on 5th Street to $y = x + 1$ (to point P).

I turn right and go east to Equality Boulevard, $y = x$ (to point Q). Then I go south, back to the x-axis (to point R). The street address is the sum $5 + 1$, that is, 6.



Tommy almost shouted, "I like your method, Ellen. The sum 5 + 1 is the avenue address of the point P. When you go east along the avenue from P, the avenue address doesn't change. At point Q (on Equality Boulevard) the street address is the same as the avenue address. When you go south from Q, the street address doesn't change. That is why the sum 5 + 1 is the street address of the point R."

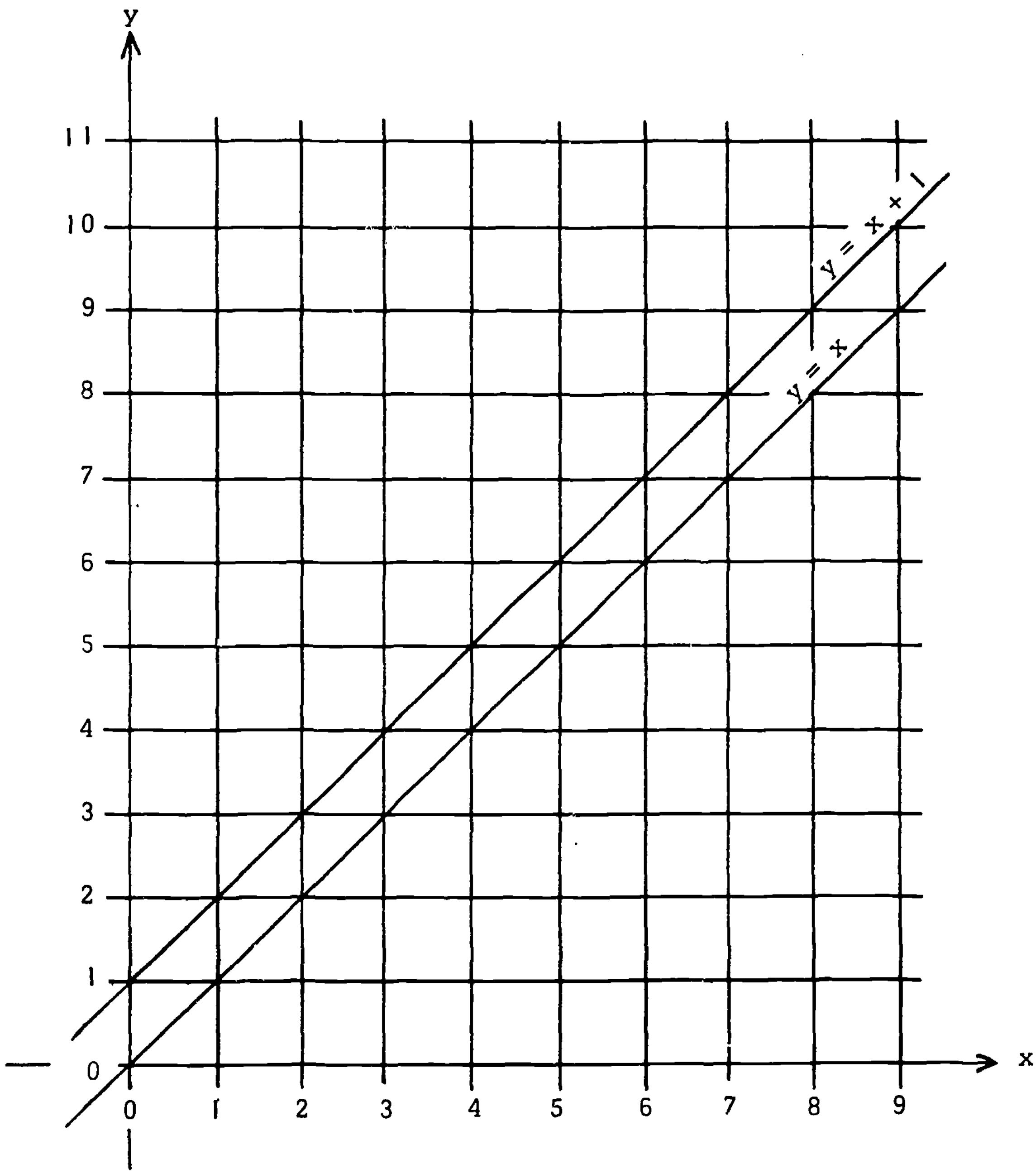
Worksheet 1 Name______________________

Use both Tommy's and Ellen's method for each problem.

Use one color to show Tommy's method. Circle his answer.

Use another color to show Ellen's method. Circle her answer.

2 + 1 + 1 = ____ 4 + 1 + 1 = ____ 7 + 1 + 1 = ____



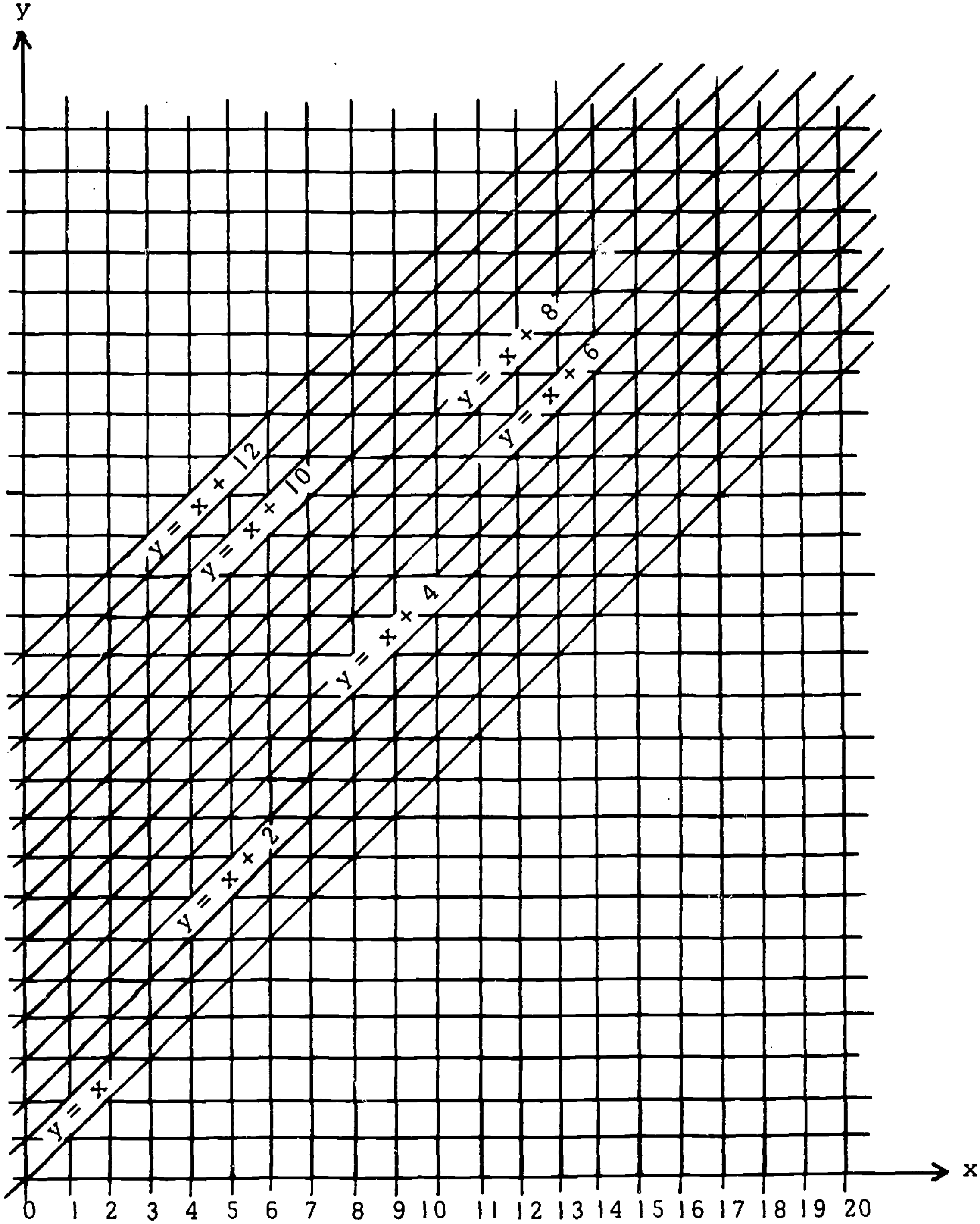
MORE ADDITION HIGHWAYS

As Mrs. Jones walked around the room checking the children's work on Squareville, she noticed something interesting. Most of the boys and girls were building new highways. When she asked them "Why?", what do you think they told her?

> A family of highways (of the form $y = x + k$) can be built north of (and south of) Equality Boulevard. These highways can be used in the same way that $y = x + 1$ is used to add 1 to any number. Why not have the children build, name, and use these highways before you go on with the story?
>
> It will be necessary for the children to use larger maps of Squareville. The idea that Squareville extends in each direction as far as we choose to extend it should be constantly emphasized.

One little fellow smiled up at her and said, "There is a highway to use when adding 1 to any number. Am I right? Well, then it would seem that we can build a highway to use when adding 2 to any number. In fact there must be a whole family of these addition highways. Here is the map I drew."

FAMILY OF ADDITION HIGHWAYS



Mrs. Jones was pleased that someone had thought of this. She knew that this family of highways would make a good adding machine. The children would be able to learn many things about addition. She was very excited as she began discussing this idea with the class.

As they were talking, Ethelbert told Mrs. Jones that he wasn't sure of what they were doing. He wondered if he might think out loud. Here is what he thought:

"When we wanted to add 1 to any number, we built the highway which is always 1 block north of Equality Boulevard. So, if we want to add 2 to any number, we just build a highway which is always 2 blocks north of Equality Boulevard. In fact, we can build a new highway for adding any number we please.

"Now how do I arrive at a name for a new highway? Maybe I should just take a street sign and plug in new numbers. Here is the street sign for Equality Boulevard:

$$Y=X$$

The highway that is always 1 block north of Equality Boulevard has this sign:

$$Y=X+1$$

"This is just a shorthand way of describing the address of every intersection on it. We found that these addresses were always

AVENUE ADDRESS = STREET ADDRESS + 1

"Suppose I want to build a highway for adding 5 to any number. I just build a highway that is always 5 blocks north of Equality Boulevard.

"It seems to me that I could make this sign for that highway:

AVENUE ADDRESS = STREET ADDRESS + 5

"In shorthand, it would be

$$Y = X + 5$$

Some children may not be able to understand this reasoning. If they don't, have them draw the highway and label the intersections. Then have them fill in this chart:

Avenue Address	= Street Address + ______
	+
	+
	+
	+
	+

Activity 2

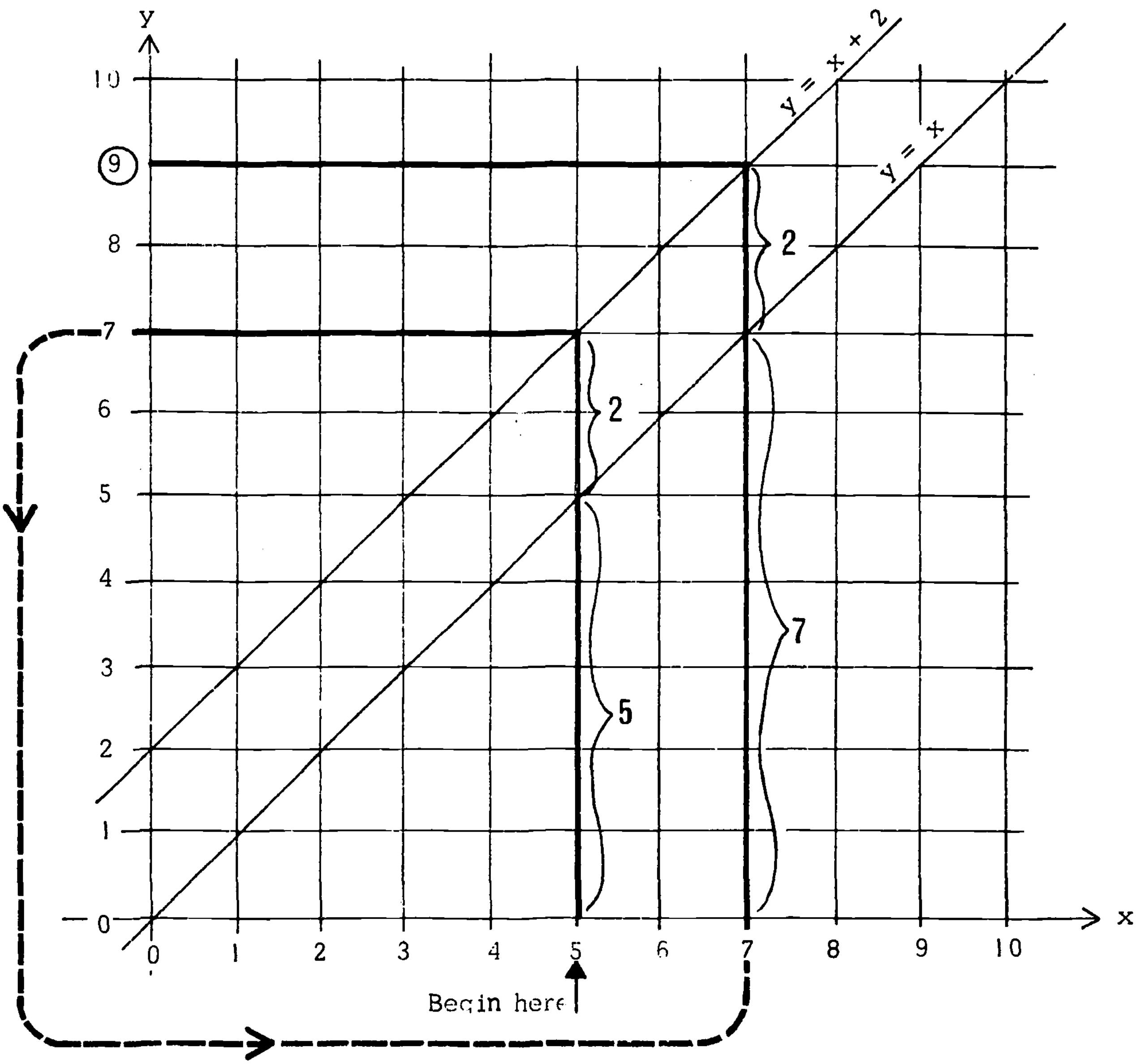
1. Give each child a map with the y = x highway marked on it.

2. Have them draw in the y = x + 2 highway.

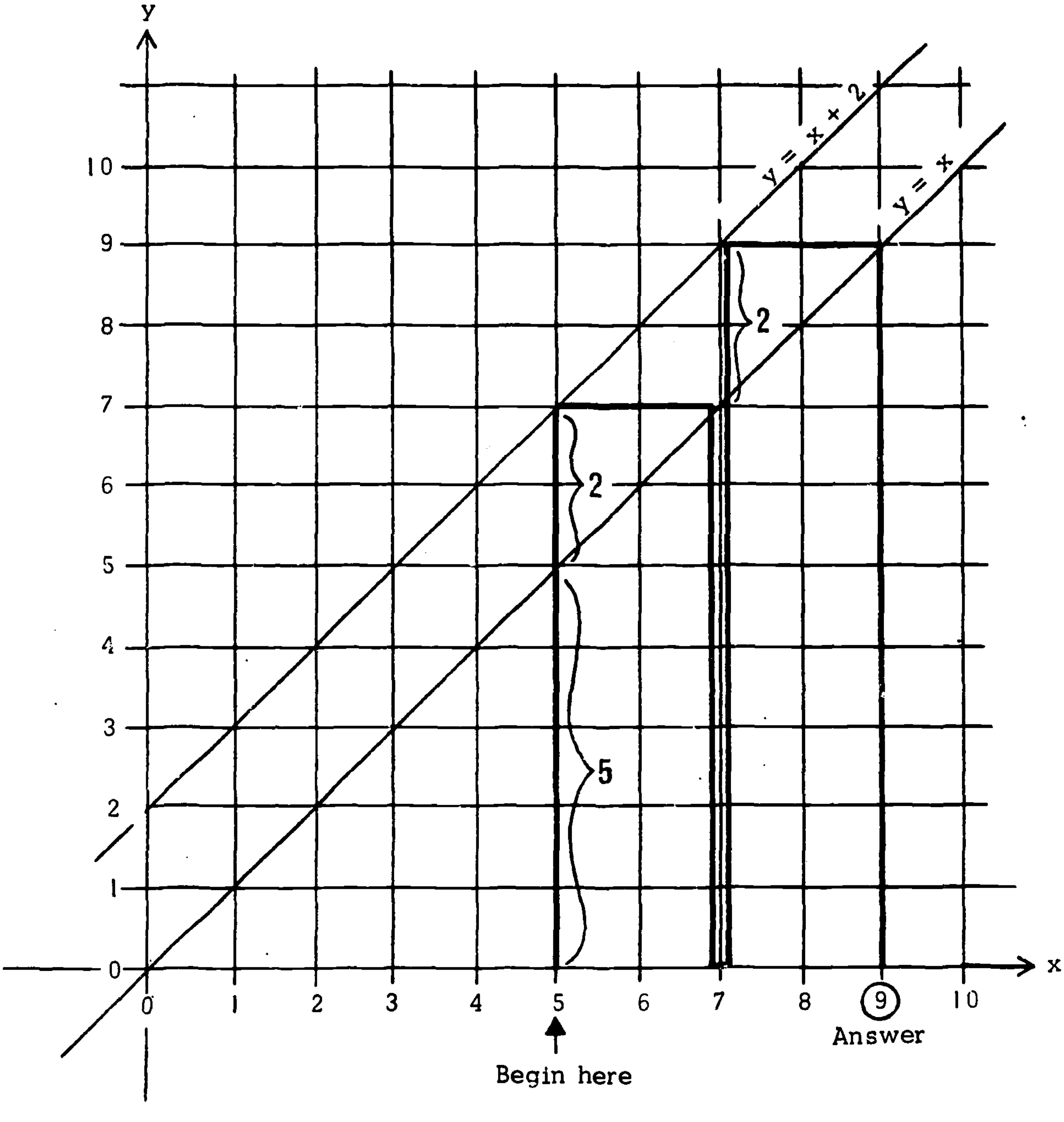
3. Then they can use either Tommy's or Ellen's method of addition in Squareville, though Ellen's is preferred. Let them work in pairs or small groups.

4. Some child might use colored chalk to demonstrate each method on the chalkboard map of Squareville.

Sample of Tommy's Method for (5 + 2) + 2 = n

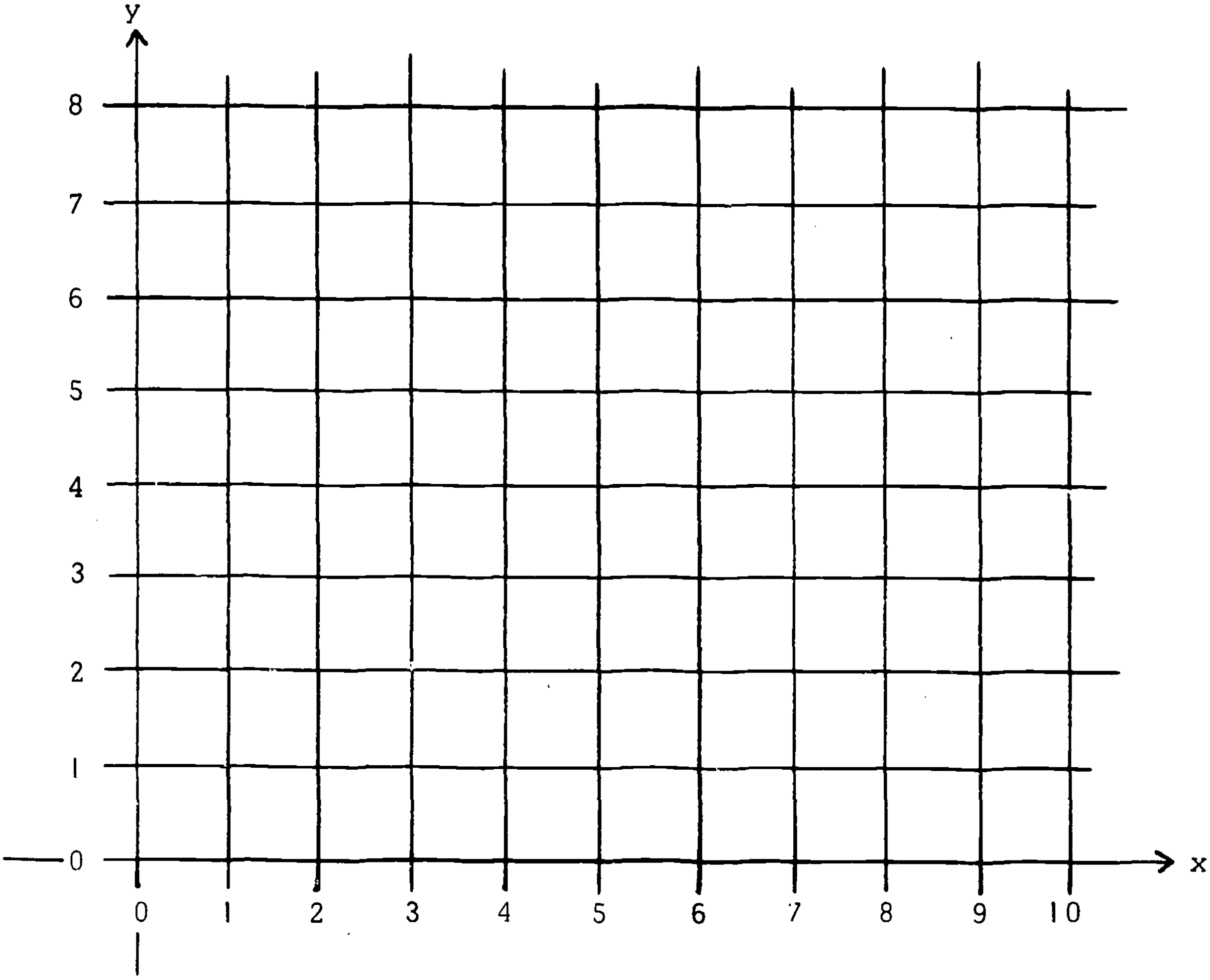


Sample of Ellen's Method for (5 + 2) + 2 = n



Make it clear to the children that the three vertical lines near S = 7 really are all on top of each other, but they are separated in the drawing for convenience.

Worksheet 2 Name______________________



Directions: Answer the first three problems on the map above and the other three on a different map that is larger. Use either Tommy's or Ellen's method. Draw the necessary highways.

1. 1 + 5 = ______

2. (1 + 2) + 3 = ______

3. (1 + 3) + 2 = ______

4. 7 + 5 = ______

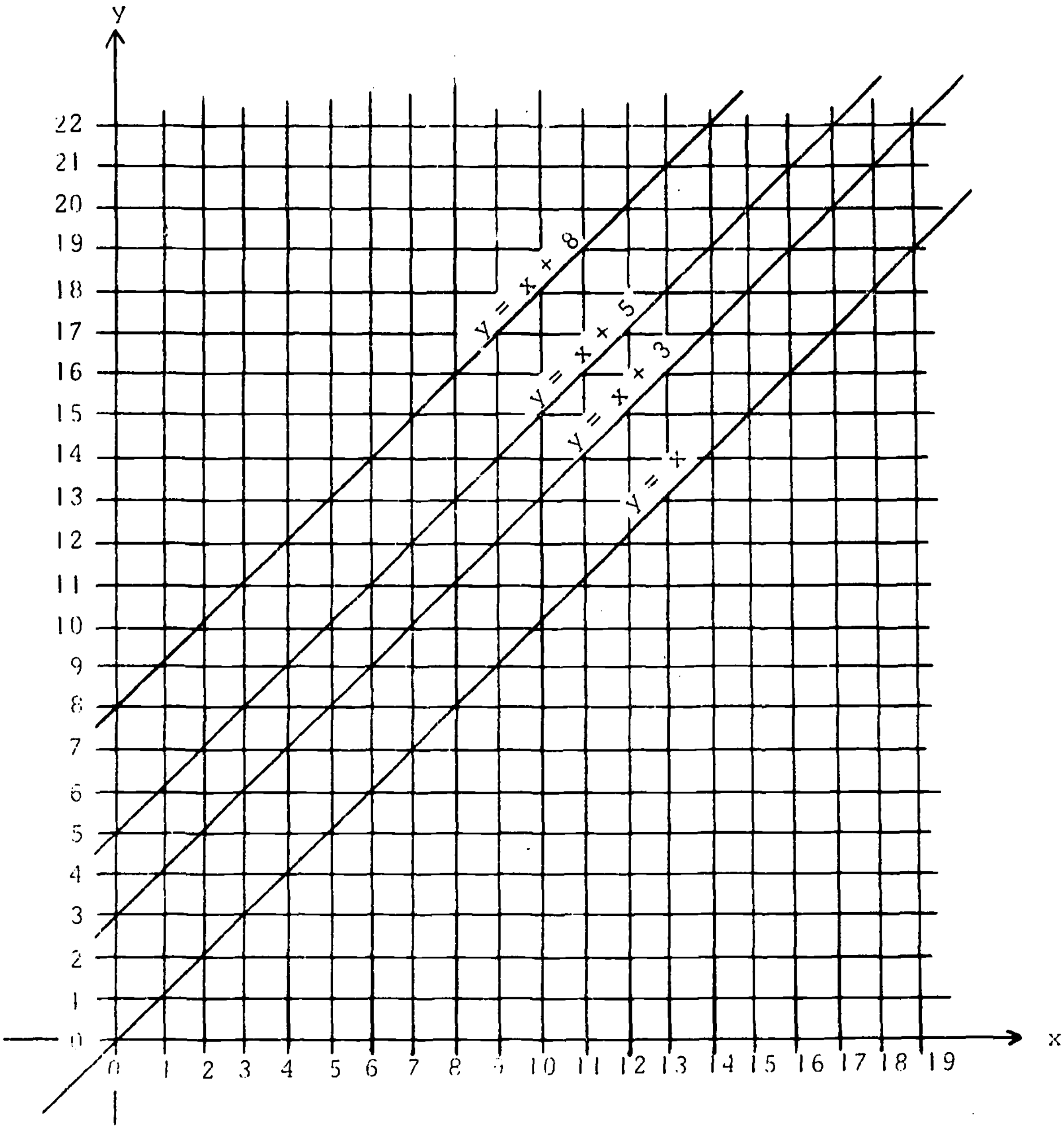
5. (7 + 3) + 2 = ______

6. (7 + 2) + 3 = ______

Did you notice anything interesting? ______________________

____

Worksheet 3 Name____________________



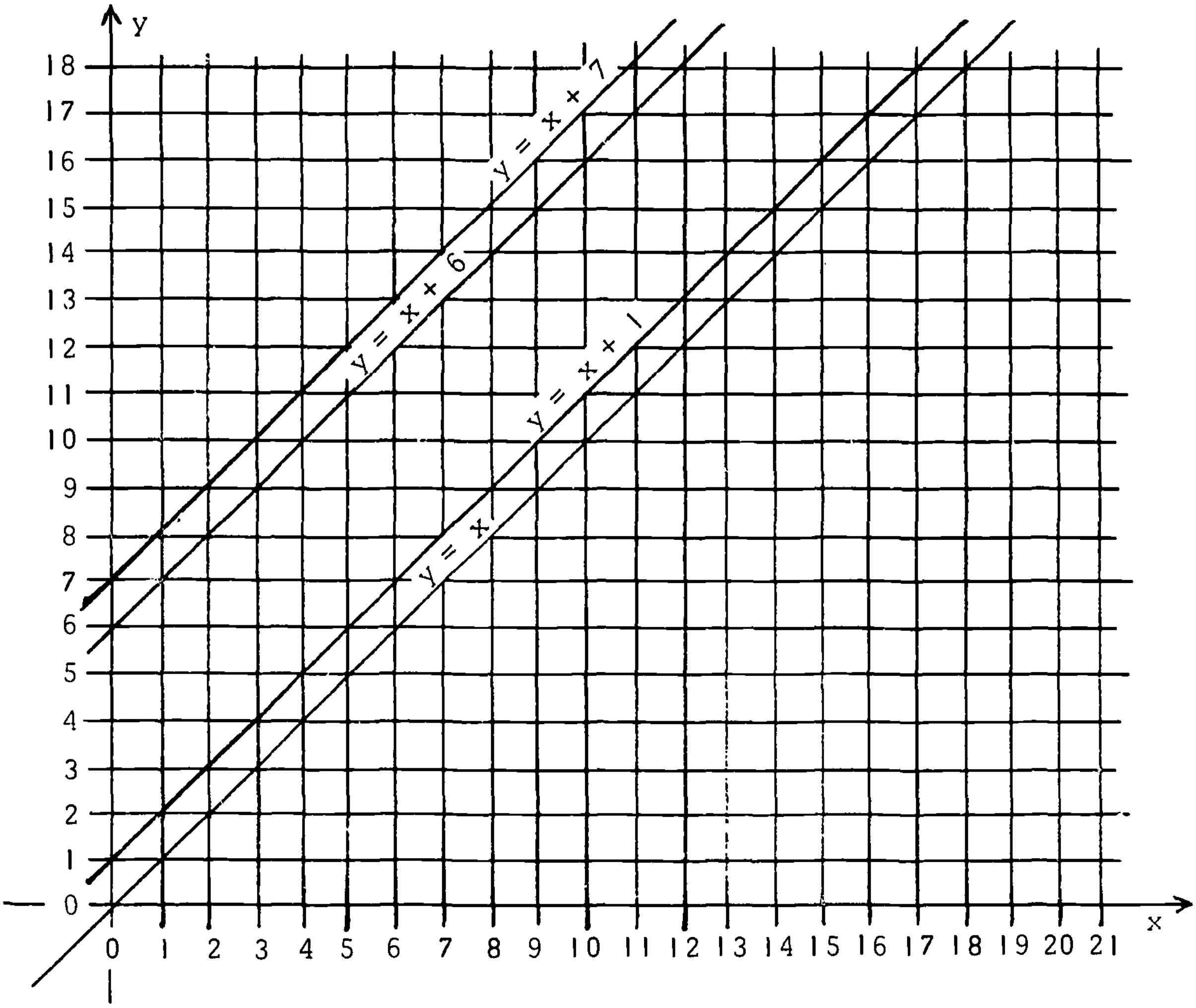
Directions: Answer the following problems on the map above or on other grids. Use Tommy's method on odd numbered problems and use Ellen's method on even numbered problems.

1. 4 + 8 = ____
2. (4 + 3) + 5 = ______
3. (4 + 5) + 3 = ______
4. 12 + 8 = ______
5. (12 + 3) + 5 = _____
6. (12 + 5) + 3 = ______

What interesting thing did you notice? ______________________________

____

Worksheet 4 Name________________________



Directions: Use Tommy's method on even numbered problems. Use Ellen's method on odd numbered problems. Get additional graph paper if you need it.

1. 2 + 7 = ________

2. (2 + 1) + 6 = ______

3. (2 + 6) + 1 = _______

4. 11 + 7 = __________

5. (11 + 1) + 6 = ______

6. (11 + 6) + 1 = ______

What interesting thing did you notice? ______________________________

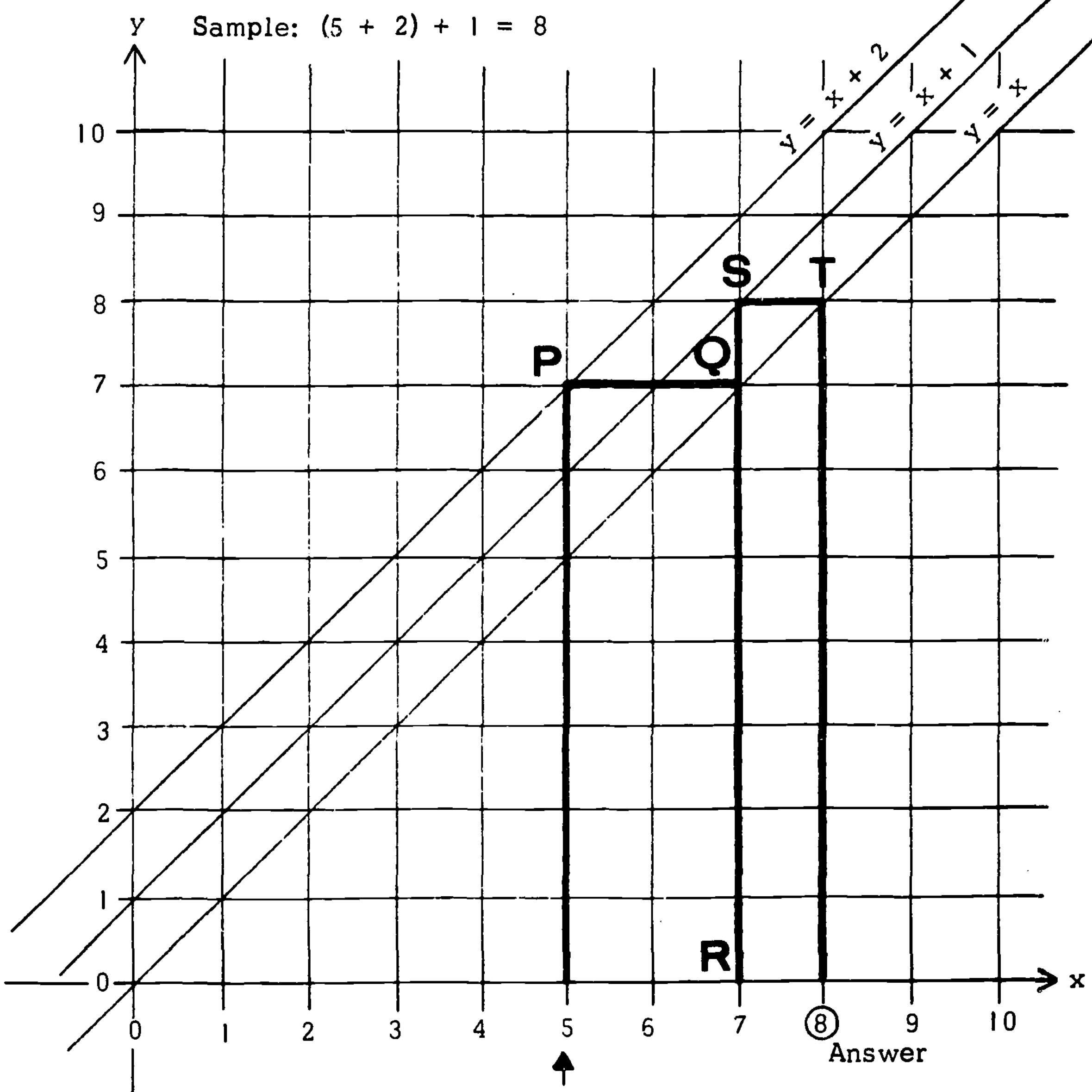
____

ETHELBERT ADDS IN SQUAREVILLE

One day Ethelbert made this clever comment to the class. "There must be a faster way to use the addition highways of Squareville. Ellen's way is faster than Tommy's, but there must be a way that is even faster than Ellen's."

Having said that, he sat back and watched, for he already knew the answer.

> Let the children suggest what Ethelbert's short-cut might be. Review Ellen's method with the class:

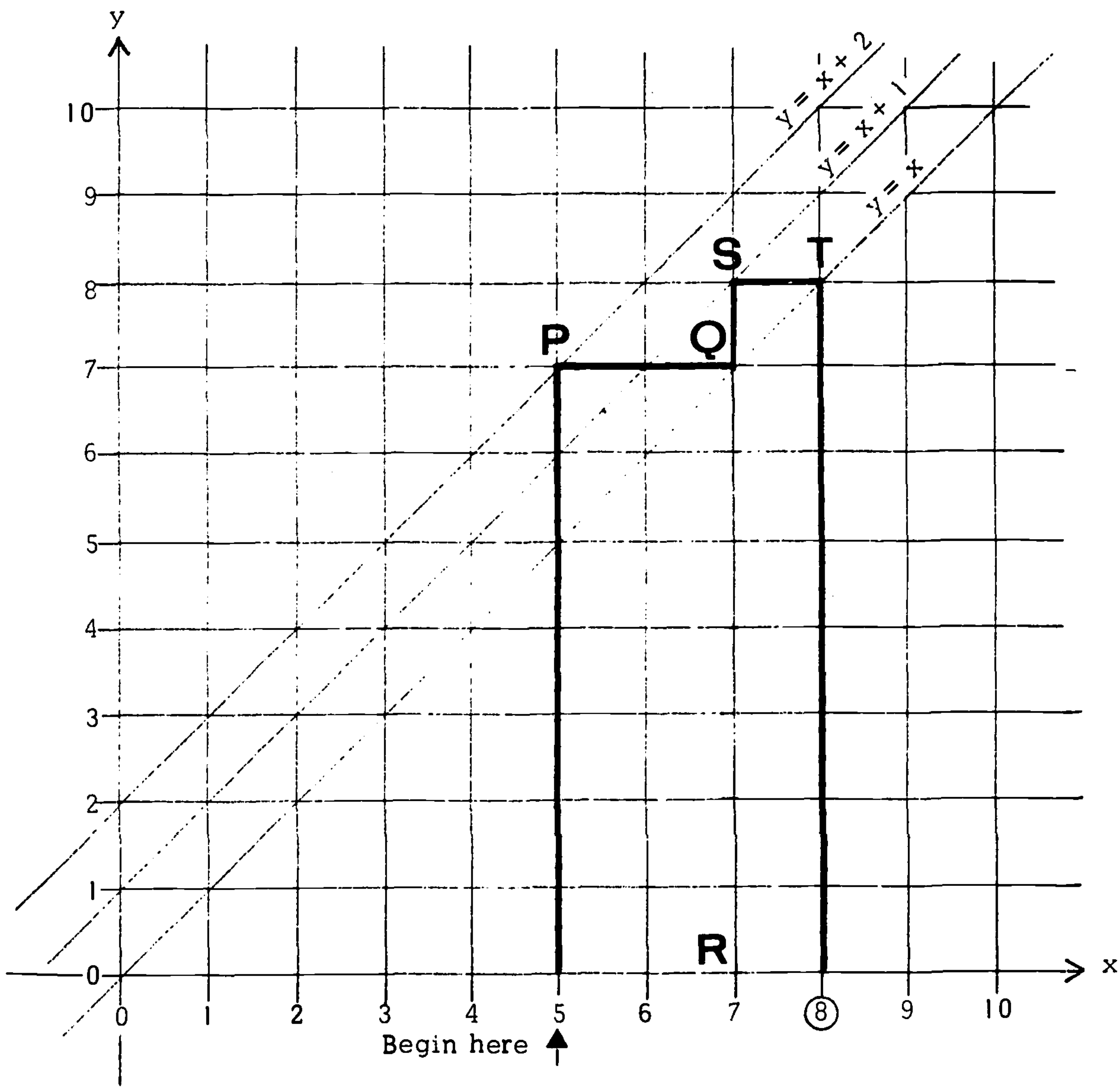


When no one came up with a new way, Ethelbert smiled to himself. He said, "Look at the way Ellen showed us how to compute (5 + 2) + 1. She started at (5,0), went north to P, east to Q, south to R, then north to S, east to T, and finally south to get the answer on the x axis.

"She had to go over the line segment between R and Q twice, once south, then north again. It really isn't necessary to do that.

"I take a shortcut. I start where she did at (5,0). When I go to Q, I go directly north to S, and skip the double trip she has to take by her method.

"From S, I go east to T, and then south to find the answer. Neat, isn't it?"



Ask the class members to draw Ethelbert's route on their maps. Have someone put it on the chalkboard. Use other problems, such as (2 + 3) + 5, to reinforce the idea.

Tommy clapped his hands excitedly and said, "That is really a neat trick for combining additions. Don't you think so, Mrs. Jones?"

She smiled in agreement and said, "That is a convenient and very powerful way of doing it, for when we're done we're ready to add again! Right now let's practice it to be sure we can all do it."

Let the children work in pairs or small groups. Provide them with maps of Squareville. After they assign problems to each other, each child will then need to draw the highways he needs. Squareville will sometimes not be large enough to contain Ethelbert's route. Let the children make suggestions for solving this dilemna (use bigger sheets of graph paper).

SUBTRACTION IN SQUAREVILLE

> Each child should have a map of Squareville. As the story is read to the children, have them record on their maps. At the same time have someone demonstrate on the chalkboard map.

One day Ethelbert said, "Here is a riddle:

I am using Tommy's method to add 3 to a number.

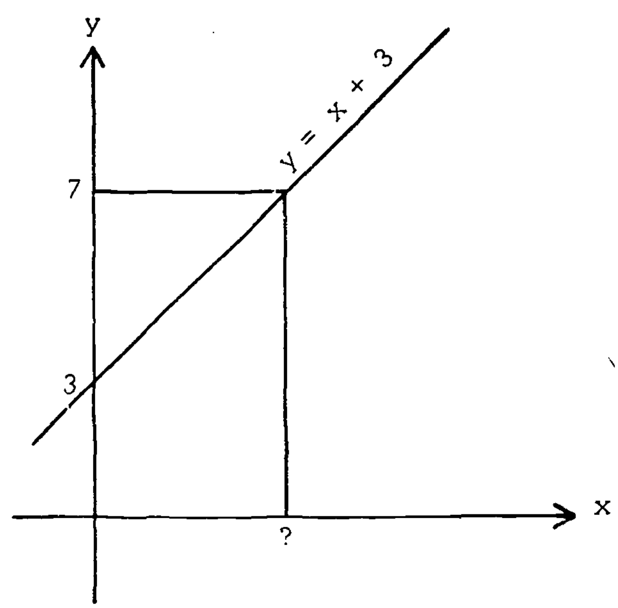
If I use highway $y = x + 3$, I will land at 7th Avenue.

Where did I start?

In other words, to what number did I add 3?"

> Explore all the solutions offered by the students.

When no one came up with a good answer, Ethelbert drew this map and asked them all to make this drawing on their own maps.



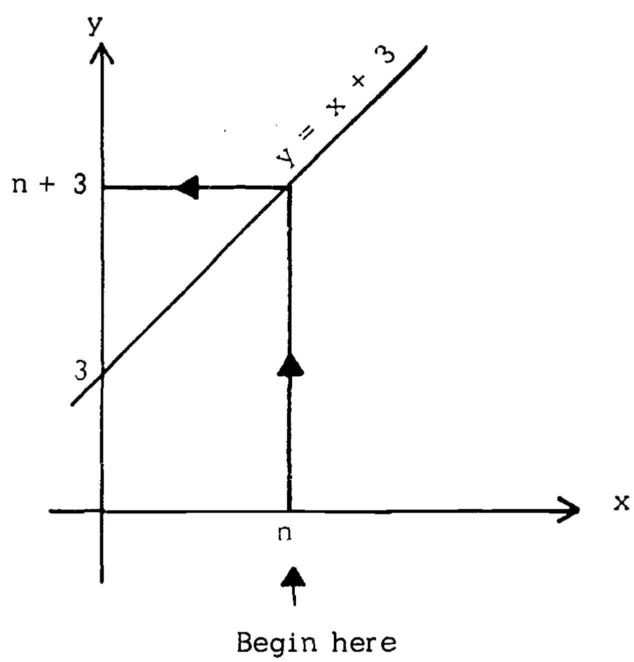
The grid has been left out in this and subsequent graphs for the sake of clarity, but if this is confusing to the children, then redraw the graphs with the grid included.

Every child did just that. Soon almost everyone had his hand up. They all had an answer to the riddle. Most of them had the answer of 4.

Tommy was eager to explain his method to the class. He said, "Ethelbert's riddle is about subtraction. All he does is reverse my method of adding.

"When I want to add 3 to some number, call it n, I begin at the point (n,0), on the x axis. Then I go north on "n street" to the line $y = x + 3$. From there, I go west on that avenue to the y axis. The avenue address at that point is the sum $n + 3$. I'll draw arrows to show my route."

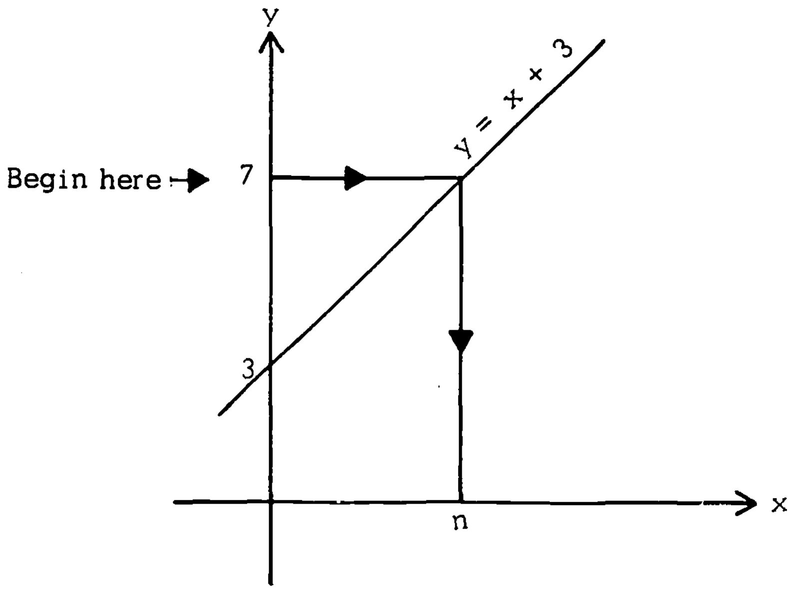
Tommy's Addition Route



Ellen laughed, "Ethelbert's answer is obvious. If you are given the sum and the number 3 you can find n by reversing Tommy's route."

She drew this picture:

Reversing Tommy's Route



Be sure that these methods are demonstrated on the chalkboard map.

Ellen went on to say, "When we use Tommy's route to solve the problem "n + 3 = ?", we use the line y = x + 3, and end up at the point (0,7) or 0 Street and 7th Avenue. The avenue address, 7, is the sum. To solve Ethelbert's riddle and find n, we will start at the point (0,7), and reverse the route. We go east to the line y = x + 3, and then south to the point (4,0) on the x axis. The answer n = 4 is the street address of this point."

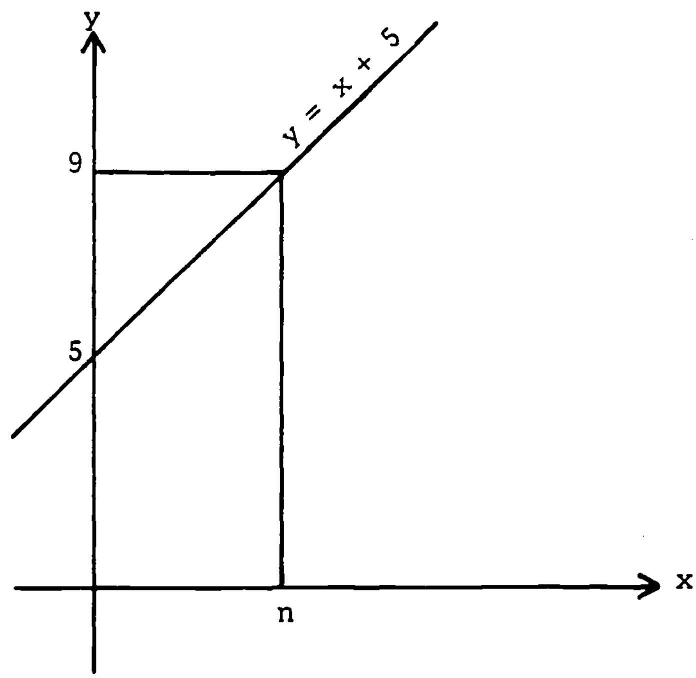
Ethelbert was delighted, "Now let's see if the others understand. Give them another problem."

> This might be a good time to let the students suggest other equations and solve them by reversing Tommy's route.

Ellen said, "I add 5 to some number n by Tommy's method. The sum is 9. To what number n did I add 5?"

Everyone agreed on the following solution:

1. When you add 5 by Tommy's method, you use the highway $y = x + 5$. This gives you your route on the map of Squareville.



Your answer is 9 because you end up at the point (9,0) on 9th Avenue.

2. To subtract, you must look for the point on 0 Street with the avenue address 9. This is the starting point. When adding you came west from highway $y = x + 5$ to this point. To subtract you do the reverse and go east from (0,9) to the $y = x + 5$ highway.

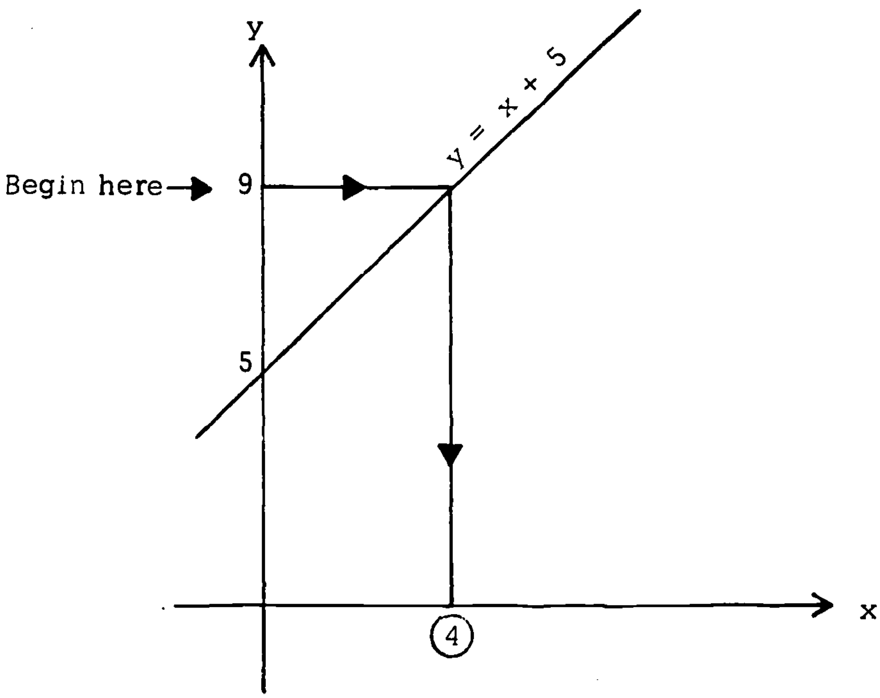
3. When adding you get to the $y = x + 5$ highway by going north from 0 Avenue.

When subtracting you reverse and go south from the $y = x + 5$ highway.

4. The answer is the street address of that point on 0 Avenue, in this case 4. 4 is the number we would have started with had we been adding.

5. You can check your answer by adding 5 to this number. The sum should be 9.

Ethelbert exclaimed, "You've got it! You've solved the equation $9 = n + 5$. You found that $n = \underline{4}$. I like your picture of subtracting 5 from 9."



Art, who had been watching closely, spoke up, "We can say that the solution of the equation $9 = n + 5$ is $n = 9 - 5$ or better $n = 4$.

> Write some equations on the chalkboard. Let n be the unknown. Have the students solve these equations by graphing them on a map of Squareville. Then have them rewrite the equations using a minus sign.
>
> EXAMPLES
>
> $10 = n + 4$ $\qquad$ $n = 10 - 4$
>
> or
>
> $9 = n + 6$ $\qquad$ $n = 9 - 6$

Art pointed out, "We can also say, in general, if y is any number, then the solution of the equation $y = n + 5$ is $n = y - 5$."

Tommy said, "Why is that?"

Worksheet 5 Name____________________

Work with a partner or in a team. Assign certain problems to each member of your small group.

For each problem, draw a certain highway on Squareville. Solve your assigned problems and show your solutions on the map of Squareville. Use as much graph paper as you need. Don't crowd your work.

Let your group decide if your method is correct.

1. Find the unknown (n).

   a) 3 = n + 2 n = ______

   b) 7 = n + 2 n = ______

   c) 12 = n + 4 n = ______

   d) 15 = n + 4 n = ______

   e) 13 = n + 5 n = ______

2. If y = n + 5,and y = 9, then n = __________

3. If y = n + 5 and y = 19, then n = __________

4. If y = n + 3 and y = 10, then n = __________

5. If y = n + 7 and y = 10, then n = __________

6. Compute the following on your Squareville map.

   a) 14 - 3 = __________

   b) 14 - 4 = __________

   c) 14 - 5 = __________

   d) 15 - 5 = __________

   e) 16 - 5 = __________

   f) 16 - 15 = __________

BE SURE TO KEEP YOUR PAPER AND YOUR MAPS. YOU WILL USE THEM AGAIN.

ELLEN SUBTRACTS IN SQUAREVILLE

One day Ellen spoke up and told the class, "If you really want to subtract quickly, you should try reversing my addition route."

Have the students try this on their maps.

Everyone did just that. Even Ethelbert could do it easily. Ellen let him explain to the other boys and girls. Here is his explanation:

"I will use my old equation, $7 = n + 3$.

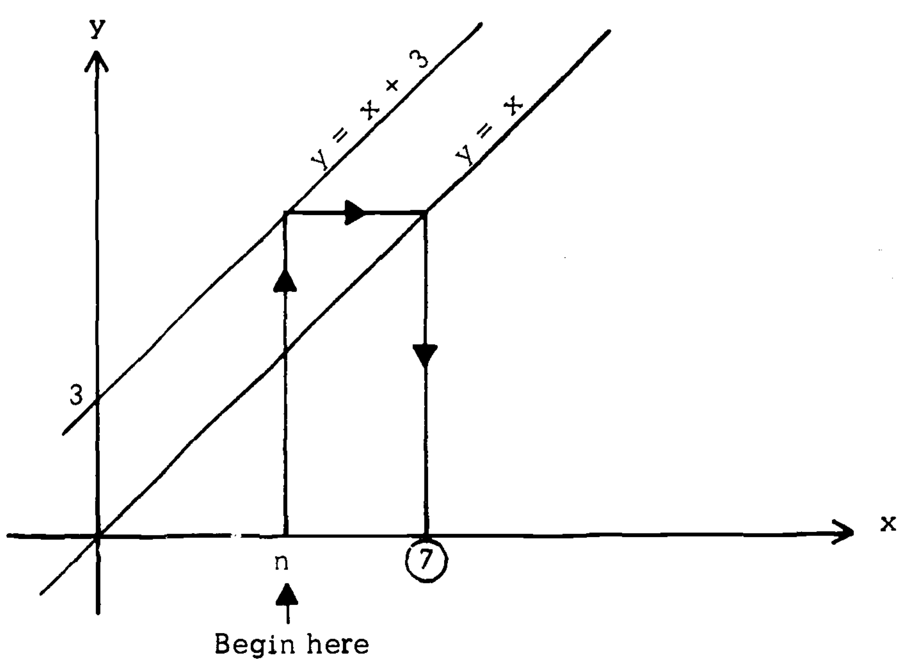
When I add by Ellen's method I must start at point n on 0 Avenue (n, 0).

Because I am adding 3, I go up n Street to highway $y = x + 3$.

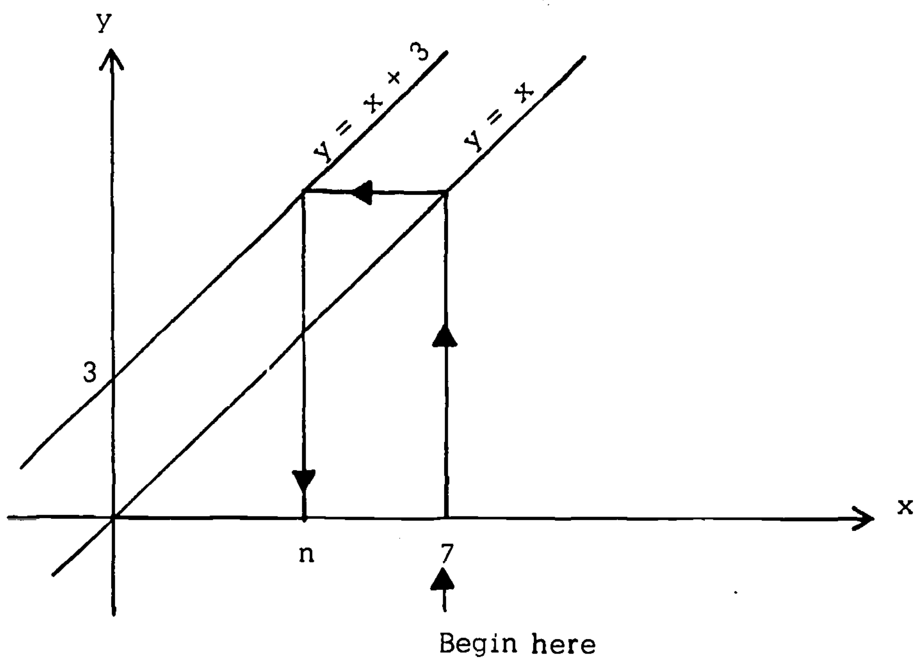
Then right to highway $y = x$, and down to 0 Avenue to the point (7,0).

The street address of 7 is the sum.

I can draw this picture of Ellen's addition route:



I can reverse Ellen's addition route and draw this picture.



"Sure enough, I can come back to point n on 0 Avenue."

> Have the children repeat Worksheet 5 using Ellen's method and the same grids they used for Worksheet 5. Encourage them to notice why both methods work.

Worksheet 6 Name________________________

Work with a partner or in a team. Assign certain problems to each member of your small group.

For each problem draw the highway in Squareville that you need and find the answer in Squareville. Use as much graph paper as you need.

Let your group decide if your method is correct. Try Ellen's method on some of the problems and Tommy's on others, as you prefer.

1. Compute the following in Squareville:
   - a) 7 - 2 = ________
   - b) 9 - 5 = ________
   - c) 12 - 12 = ________
   - d) 12 - 9 = ________
   - e) 22 - 4 = ________

2. Solve the following equations:
   - a) 9 = k + 2 k = ____________
   - b) 14 = 3 + k k = ____________
   - c) 6 = k + 8 k = ____________
   - d) k = 17 - 13 k = ____________
   - e) 22 = p + 7 p = ____________
   - f) r = 21 - 17 r = ____________

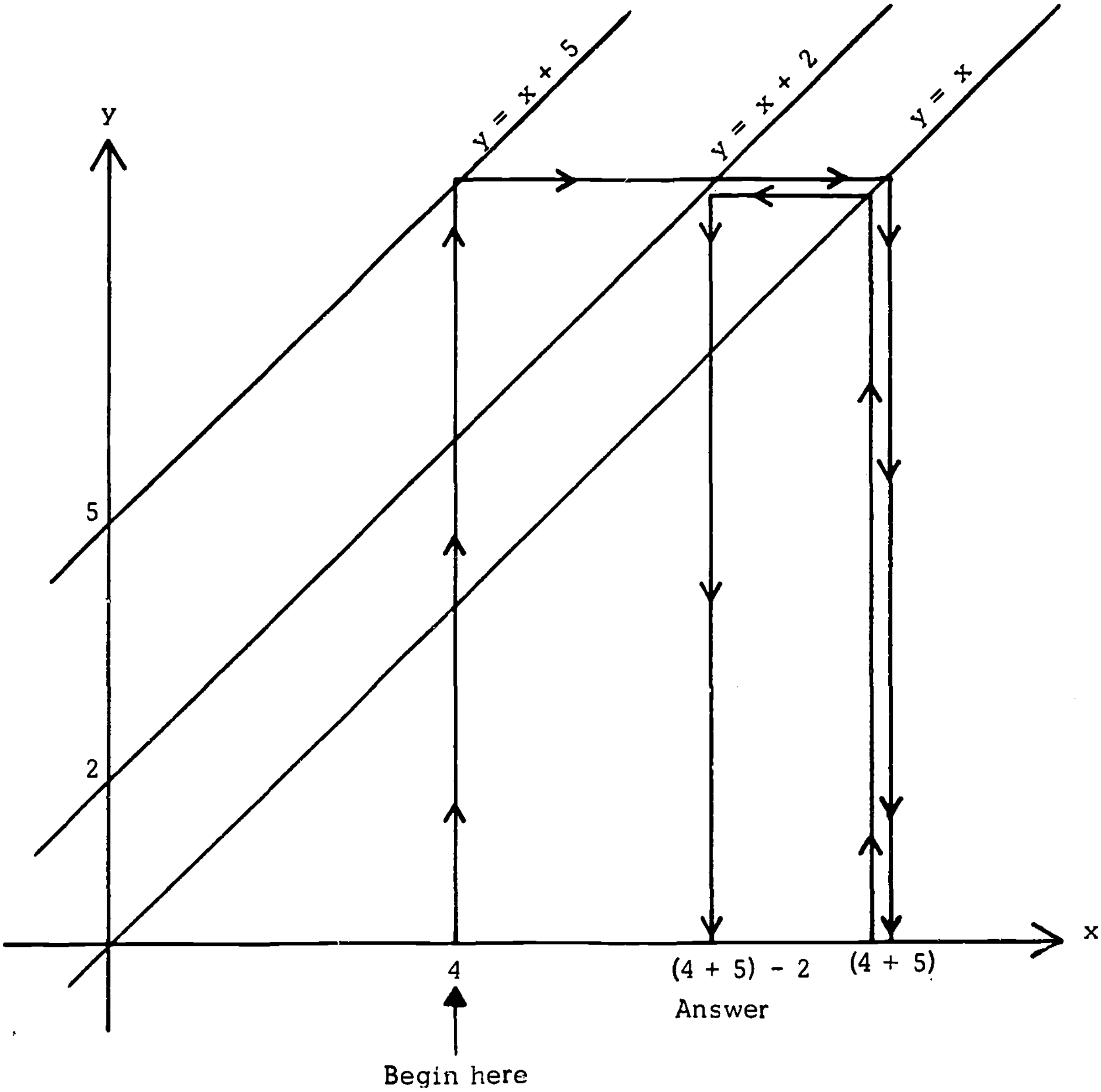
ADD AND SUBTRACT IN SQUAREVILLE

One day Ann was busy solving equations. Suddenly she commented to Mrs. Jones, "It doesn't matter whether I add or subtract. I have to do about the same amount of work for both operations and they are really almost the same thing. I wonder if it is possible to work problems that require two operations."

"Let us try it," said Mrs. Jones. "I'll write some problems on the chalkboard. You try to use both Tommy's and Ellen's methods, maybe one will be faster than the other."

> Work these problems by Ellen's method. Let the children demonstrate their methods. Compare their routes on Squareville.

Example: (4 + 5) - 2



Ethelbert's shortcut in addition can be applied by going directly right from y = 5 to y = x and then back to y = x + 2.

Worksheet 7 Name________________________

Work with a partner or in a team and solve these problems. Remember to first work out the problems in parentheses. For example, when we solve (3 + 4) - 2 = ? , we first add 3 and 4, and then subtract 2.

Show your work on Squareville grids.

1. a) (3 + 4) - 2 = ________

   b) (10 + 3) - 2 = ________

   c) (14 + 3) - 7 = ________

   d) (10 + 3) - 9 = ________

2. a) (5 - 2) + 1 = ________

   b) (10 - 3) + 5 = ________

   c) (10 - 5) + 3 = ________

   d) (5 - 8) + 7 = ________

3. a) (13 - 3) - 4 = ________

   b) (17 - 8) - 1 = ________

   c) (14 - 5) - 3 = ________

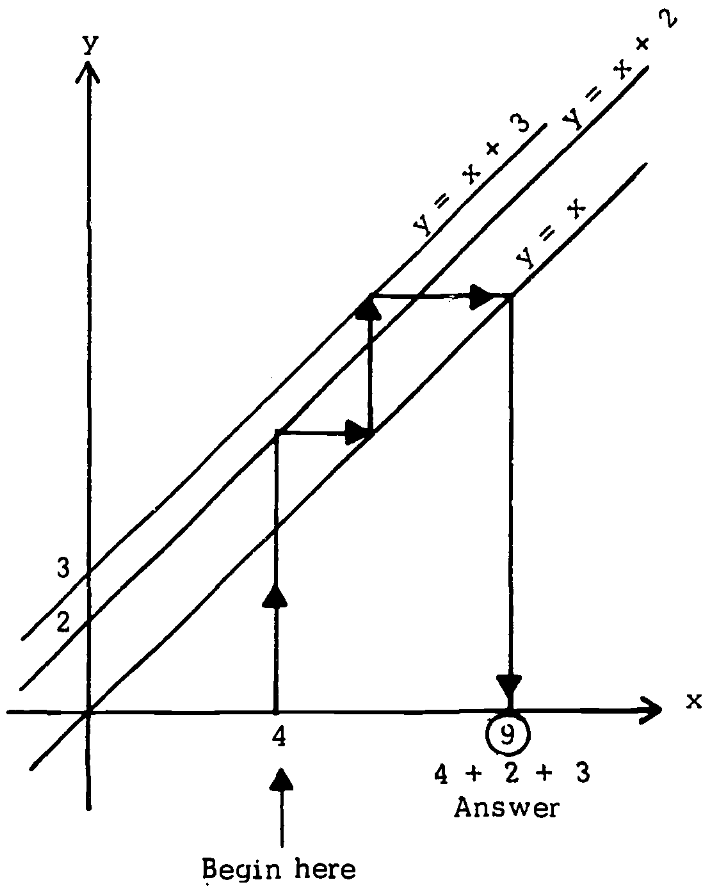
   d) (7 - 1) - 6 = ________

Tommy became very excited. "Ethelbert's shortcut should work on these problems too. His route must be faster than either Ellen's or mine. Let's try using his route."

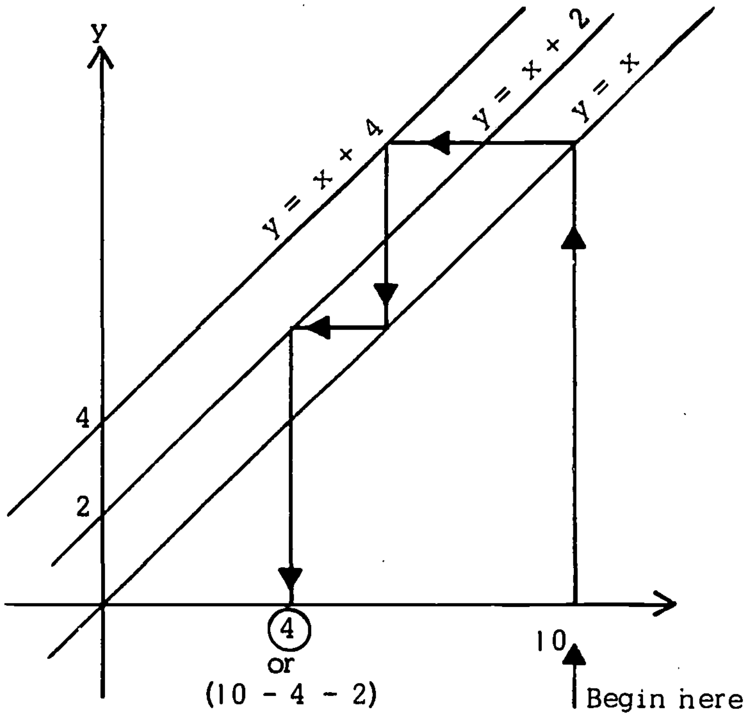
> If some child has discovered this, let him demonstrate. If no one has discovered it, suggest that they recall Ethelbert's method of addition.

Tommy reminded the children of how Ethelbert had added three numbers such as 4 + 2 + 3.

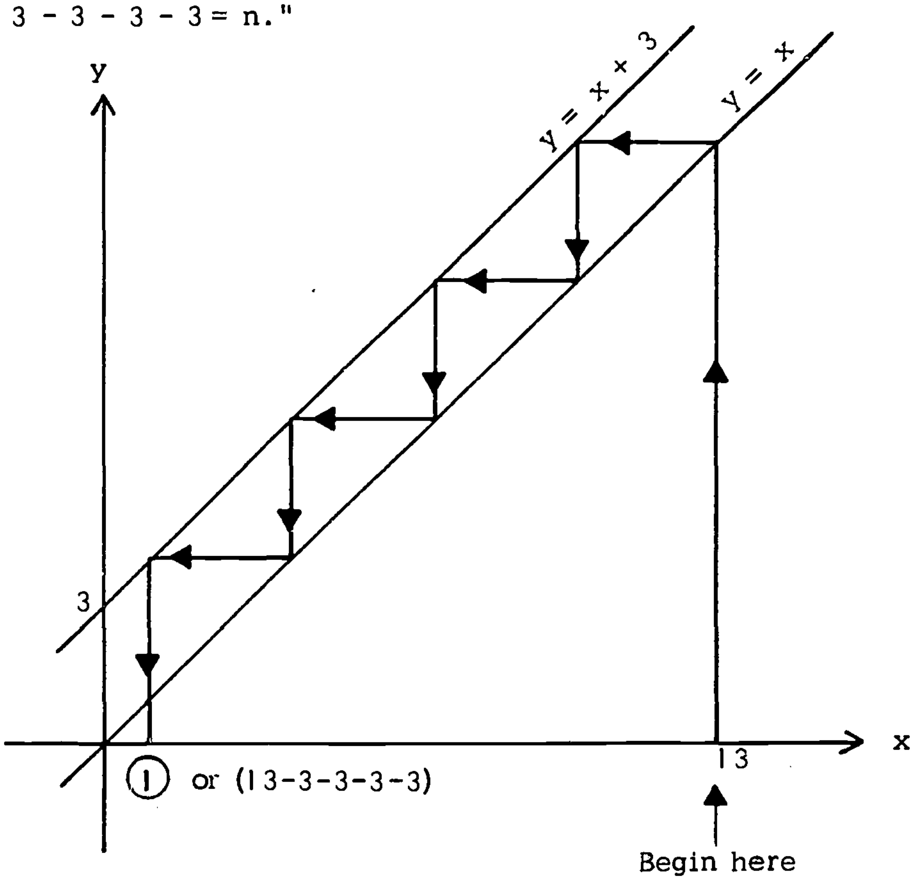
Ethelbert's Method



Then he suggested that they think of this equation, $(10 - 4) - 2 = n$.
Tommy felt that Ethelbert's solution might be:



Ellen said, "Ethelbert's shortcut is a good method to use with problems like $13 - 3 - 3 - 3 - 3 = n$."



Pose additional problems to the children here if they need additional practice, or ask them to repeat some of the problems in Worksheet 7 using Ethelbert's method.

Mrs. Jones commented, "If you want to see something interesting, solve these two equations:

$$(5 + 2) - 2 = \_\_\_\_\_$$

and

$$(5 - 2) + 2 = \_\_\_\_\_$$

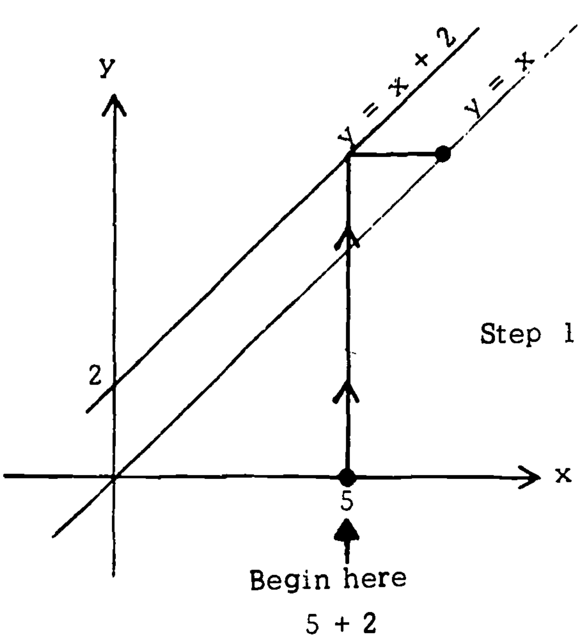
Let the children demonstrate their drawings for these problems. The generalization associated with this situation is best explained by the following equations:

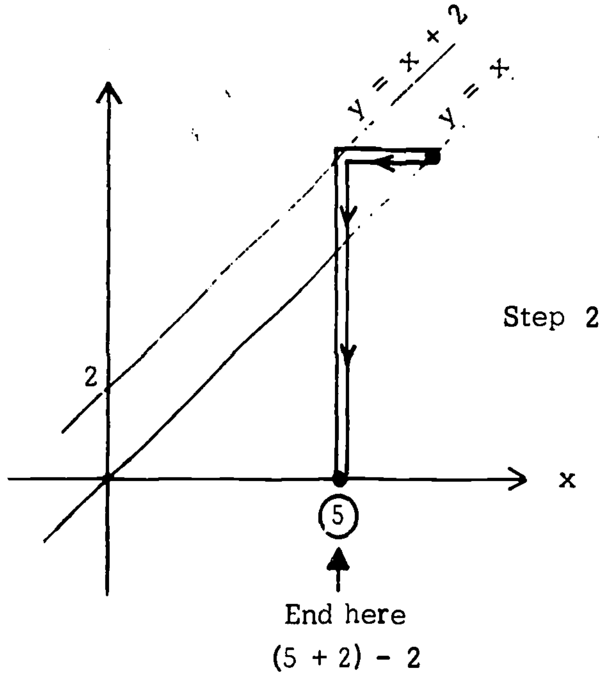
$$(x + y) - y = x$$

$$(x - y) + y = x$$

A child might say that adding a number on and then subtracting it off again has no effect on the final answer. Be sure to have the children try different numbers besides 5 and 2 to see the above relationship.

Here is how the class solved the problems by using Ellen's method. They could also have used Tommy's method.

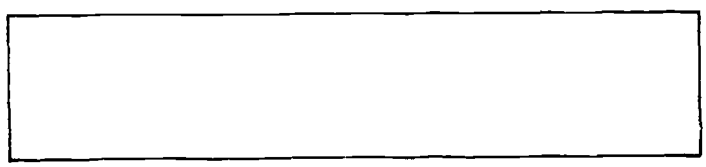




They all agreed that the answer for these equations was the same as the number they started with.

Ethelbert reminded the class of how they had named the highways in Squareville. He suggested that there must be a shorthand way of showing their discovery.

Someone then suggested that they try the sign method again. Everyone was interested, so Mrs. Jones began by drawing a large blank sign on the chalkboard.



"What shall we write on the sign?" she asked the students. "This time let's start with a long wordy sign."

Solicit suggestions as to the contents of the sign and then try to lead the children to a shorthand conclusion. Be sure to read the next part of the story before you hold a class discussion. Choices of names and letters in the story are arbitrary. Students may want to use different names and letters. Let them be original.

Tommy suggested this sign:

A First Number + A Second Number - The Second Number = The First Number

Ellen thought this was fine but she felt there should be another sign like this:

A First Number - A Second Number + The Second Number = A First Number

Tommy agreed with her that two signs were necessary. They decided to find a shorthand for both signs.

Ethelbert reminded them that they could let different letters represent different parts of the sign. He went to the chalkboard and made this new sign beneath Tommy's sign:

Ethelbert's Way of Writing Tommy's Sign

F+S−S=F

Ethelbert's Way of Writing Ellen's Sign

$$F - S + S = F$$

All the children agreed that this was an excellent method. They now needed only two short signs.

About this time Mrs. Jones asked them if they really needed two signs. She suggested that they should use parentheses to show what was really happening. She wrote this statement:

$$(f + s) - s = f$$

Then she wrote this statement:

$$(f - s) + s = f$$

Worksheet 8 Name_______________

Solve these problems in Squareville.

a) (5 + 4) + 3 = ________

b) 5 + (4 + 3) = ________

Remember to work out the steps that are in parentheses first.

b) (17 + 3) + 8 = ________

17 + (3 + 8) = ________

c) (7 + 8) - 3 = ________

7 + (8 - 3) = ________

What do you notice? ____________________

____

This property is called associativity. It lets us put in or take out parentheses when addition and subtraction are involved. This helps us solve many problems.

Name____________________

2. Solve these problems in Squareville.

a) 7 + 5 = ________

5 + 7 = ________

b) 13 + 4 = ________

4 + 13 = ________

c) 7 + 2 + 14 = ________

2 + 14 + 7 = ________

d) 8 - 3 + 6 = ________

8 + 6 - 3 = ________

What do you notice? ____

____

This property is called commutativity. It lets us switch the order of the members when addition and subtraction are involved. It is one of the very nice properties of addition and subtraction.

"Now," she said, we have to notice something very nice about addition and subtraction of numbers."

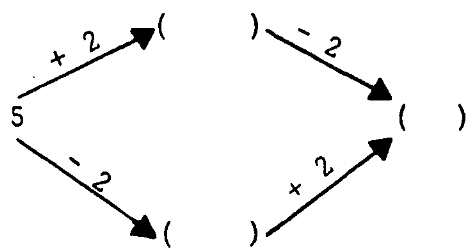
Tommy remembered that addition is associative and commutative. By this, he meant that you could write all of these statements and more.

$$(f + s) - s = f$$
$$f + (s - s) = f$$
$$(f - s) + s = f$$
$$f + (-s + s) = f$$

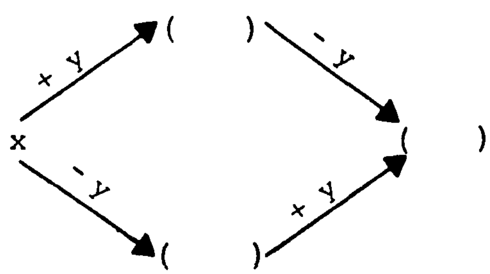
So he told the class that all they really needed was one statement such as:

$$x + y - y = x$$

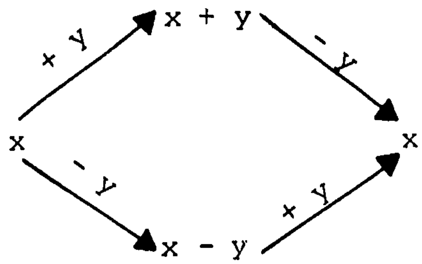
"I see it in a different way," Ellen said. "I picture the problem like this. Write it down and work it."



Everyone found that the final answer was 5. Ellen then substituted letters for numerals.

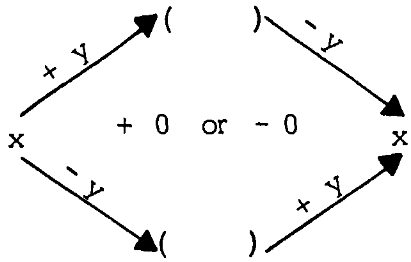


Their answer looked like this:



Tommy spoke up, "The y that you add, and the y that you subtract cancel each other. That is like adding 0 to x. You will get x for the answer."

He wrote these statements:



ALICE SUBTRACTS WITH MIRRORS

Has anyone ever told you that your ideas weren't quite as good as you thought they were? How did you feel? Did you resent it, or did you want to know how you could improve? Let us see what Ethelbert did when Ellen's cousin, Alice, told him that she could improve his method of subtracting on Squareville.

It all happened one day when Alice came to visit the class. She sat patiently while everyone proudly told her about Squareville. When they finally asked her for an opinion, she startled Ethelbert by saying, "Your way of adding and subtracting is excellent but there is another way to do it. I'll give you a hint," and she wrote this on the chalkboard:

> If a mirror is used, the mirror reflection will be the ordinary form of the name
>
> *Alice*
>
> If a mirror is placed along the line $y = x$, the reflection of the addition highway will represent the subtraction highways. If possible, demonstrate this section with actual mirrors.

"I get it," shouted Tommy. "You have written your name backwards. We can use a mirror to reverse your name so that we can recognize it."

"Are you suggesting that we use mirrors to do subtraction?" asked Ethelbert. "How can you draw routes on a mirror?"

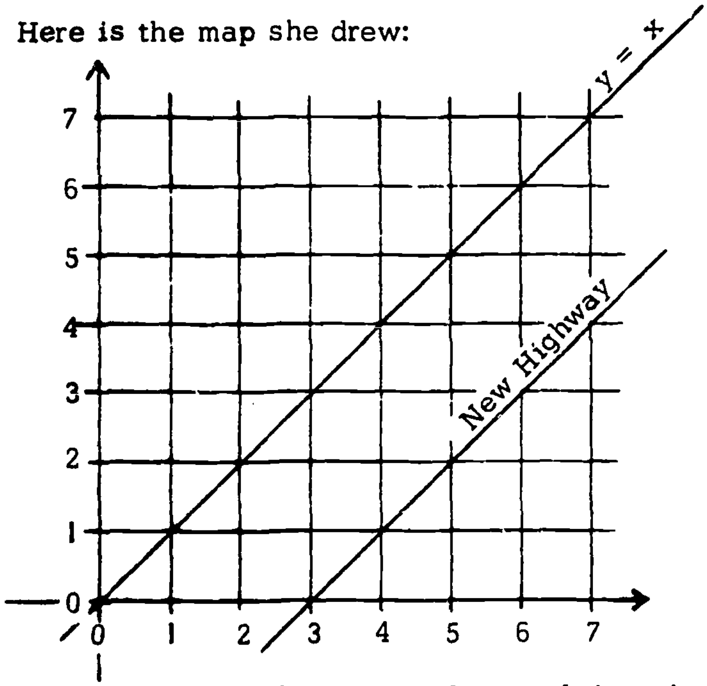
"What I mean," interrupted Tommy, "is that we do addition on highways that are north of Equality Boulevard. Why can't we build some highways south of Equality Boulevard and do subtraction on them?"

> Maybe some child will suggest that highways drawn south of $y = x$ represent reflections of the highways drawn north of $y = x$. Use mirrors to actually show this.
>
> Fold a map of Squareville along $y = x$. Then use a pencil to trace over the addition highways with enough pressure to make a crease in the reflected half of the street.

"Let's try it," Ellen said, and she began to construct a highway that was always the same number of blocks south of Equality Boulevard.

> Have some child construct a highway that is always the same distance south of Equality Boulevard.
>
> Label the intersections on the new highway.
>
> Name the new highway. The street sign approach will be helpful in naming the highways; $y = x + 3$ represents a highway that is always 3 blocks north of Equality Boulevard. $y = x - 3$ represents a highway that is always 3 blocks south of Equality Boulevard.

Here is the map she drew:



Tommy examined her map and agreed that she had done a good job. The new highway was indeed "always the same distance south of Equality Boulevard". This highway was three blocks south of y = x.

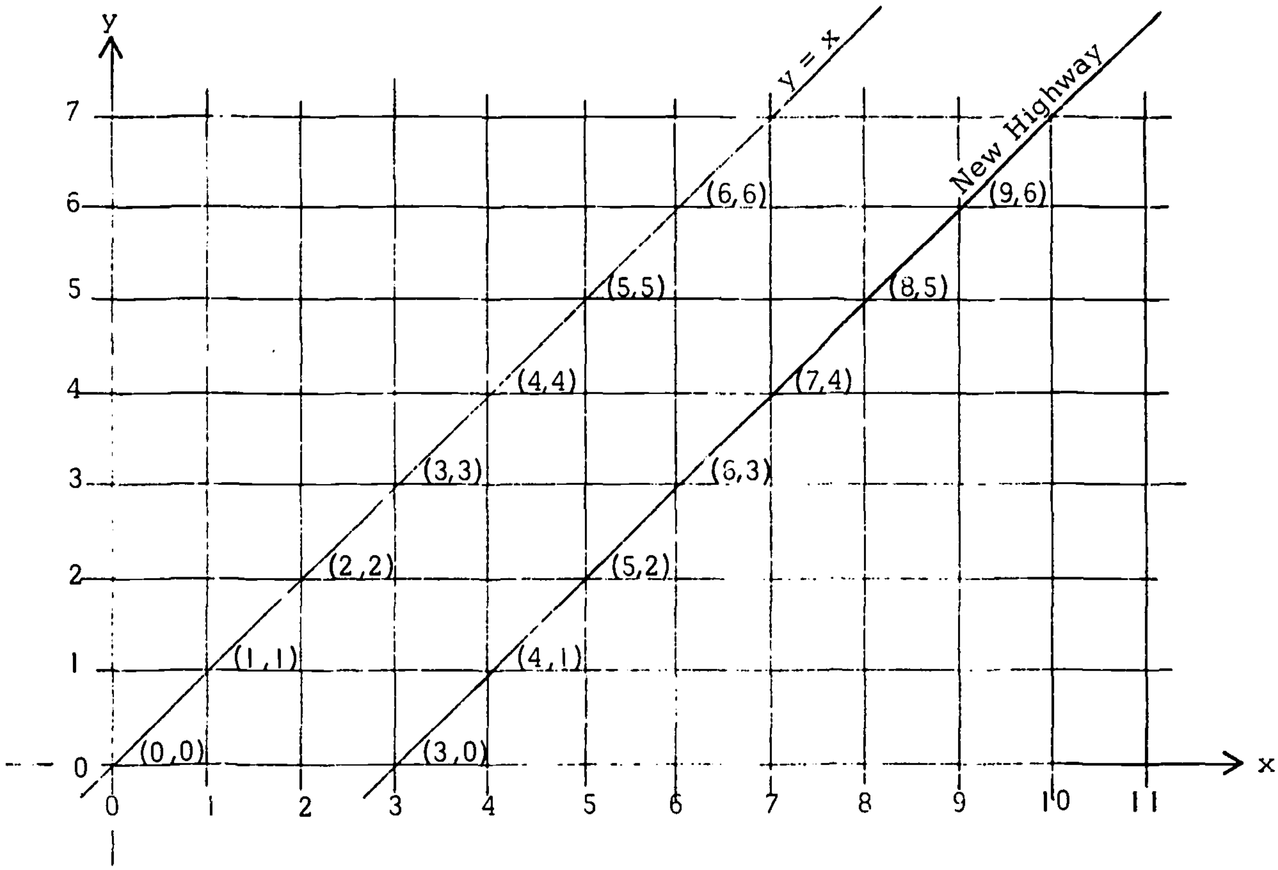
Ethelbert wondered if other highways could be built in a similar manner. Tommy told him that there was only one thing that mattered in this discussion. First you decided how far south of y = x the new highway would be. Then you built it so that it was always that far from y = x.

About this time, Alice asked Tommy to name the new highway that Ellen had drawn.

This he could do easily. He decided to first name the intersections on the new highway.

If this hasn't been done by the students, have them do it now.

Here is how they appeared:



Ethelbert offered to build this chart of the intersection addresses:

Avenue Address	Street Address + ____
0	3
1	4
2	5
3	6

Things were puzzling to him so he drew a large sign and tried to fill it in:

AVENUE ADDRESS = STREET ADDRESS ___________

His system didn't seem to work. When he added a number to the street address he didn't get the avenue address.

$$4 = 7 + ?$$

$$2 = 5 + ?$$

Suddenly he decided to subtract. This did work.

$$4 = 7 - ?$$

$$2 = 5 - ?$$

He filled in his sign with the long words:

AVENUE ADDRESS = STREET ADDRESS - NUMBER OF BLOCKS SOUTH OF EQUALITY BOULEVARD

He substituted letters and numbers and made this sign:

$$Y = X - 3$$

"That is a good name," said Alice. "Now use that highway to subtract."

Let the children suggest that this would be a usable highway for subtracting 3 from some number. The possible routes are suggested in the story which follows.

"Just watch me," said Ethelbert. "It might help you to follow me on your map of Squareville."

"Suppose I want to subtract 3 from 7,

I begin at the point on 0 Avenue whose street address is 7.

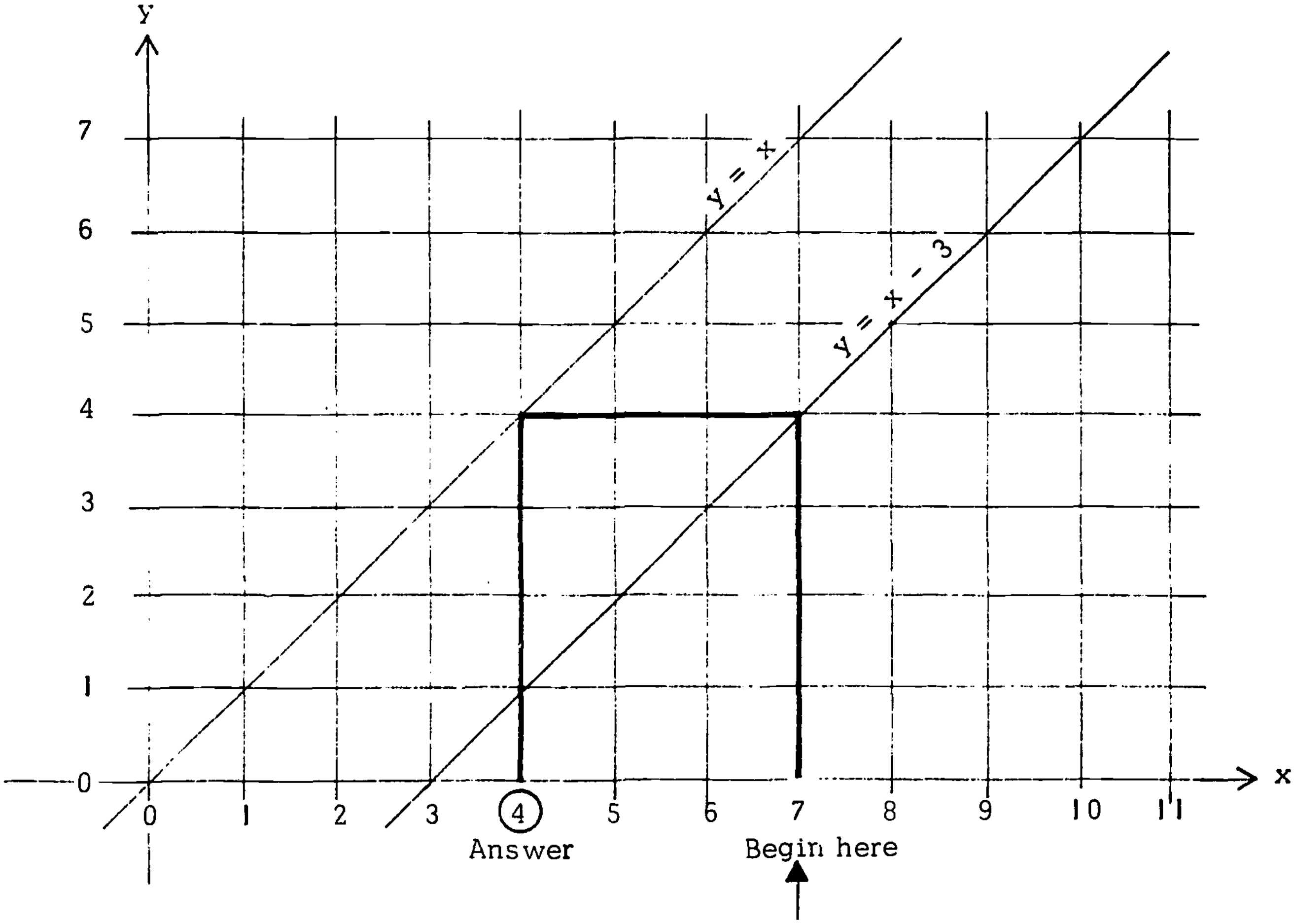
You also begin here when you want to add 3 to 7.

When I subtract, I go north only to y = x - 3.

Then I go west to Equality Boulevard, and south to 0 Avenue again.

My answer is the street address of the point (4,0) on 0 Avenue.
So 7 - 3 = 4."

We are really using Ellen's method of addition here, to add the number - 3.



"I notice something else about y = x - 3," offered Tommy. "It is a mirror image of another highway. Can you tell me the name of that highway?"

y = x + 3 and y = x - 3 are mirror images of each other. This can be demonstrated by drawing y = x, y = x + 3, and y = x - 3 on a map of Squareville. If the map is folded along the y = x highway, the y = x + 3 and y = x - 3 highways will fall onto each other. This can be seen if the paper is held up to some bright source of light.

Ethelbert was quick to answer, "I have a way of finding out. First I fold the map along Equality Boulevard. Then I hold the paper up to a window or another light source. I see that y = x + 3 and y = x - 3 fall upon each other. This shows that they are symmetrical with relation to the y = x line."

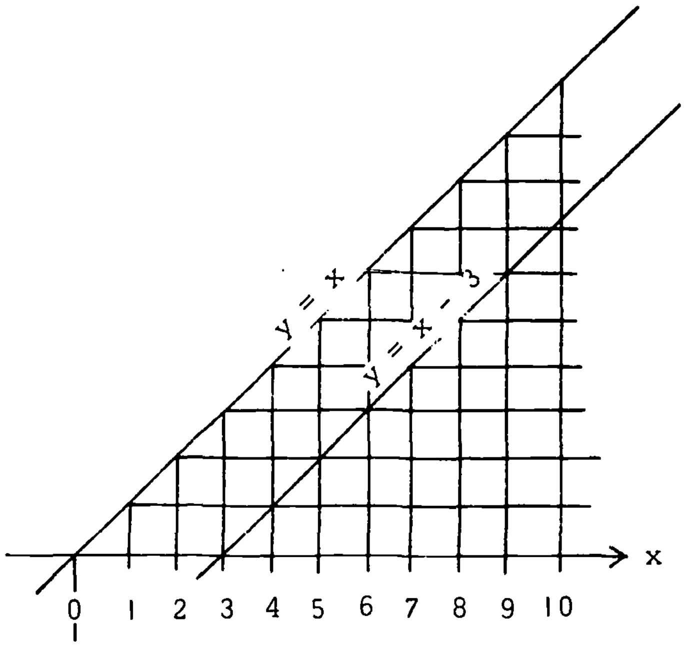
Two other ways to illustrate this are:

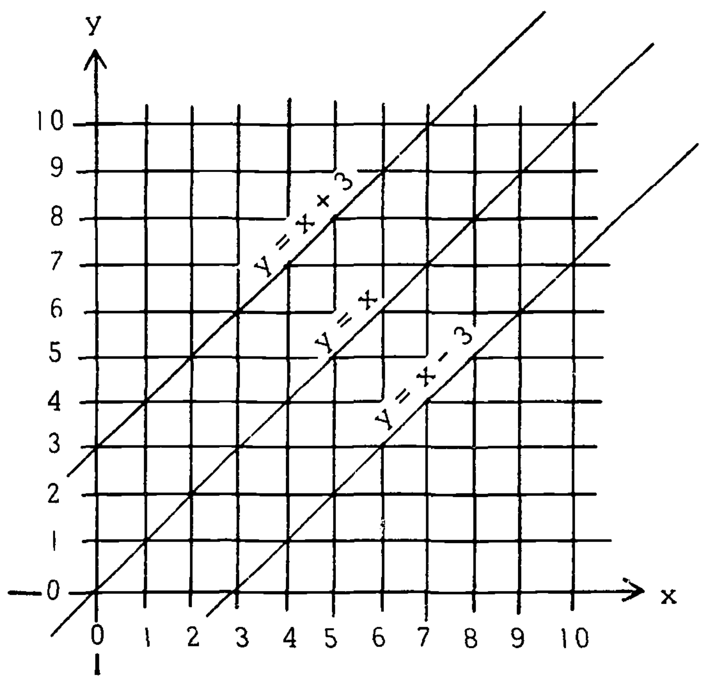
1. Fold the paper on y = x. Draw a heavy line along an addition or a subtraction highway. If you press hard enough, you will make a groove in the other half of the paper. This groove should fall on the highway that is the mirror image of the one over which you traced.

2. Select a highway other than y = x. Mark the intersections along this highway. Fold the paper along y = x. Then poke a sharp object through the paper at each intersection you marked. This will make holes on the other side of y = x. Connect these dots with a straight line and you have the mirror image.

Ellen felt that she had a good idea, too. She suggested that you choose a highway that was either north or south of y = x. Then she folded the paper on y = x. She drew a heavy line along her chosen highway. This left a groove in the paper. The groove showed the route of the mirror image for her highway.

These pictures show how she did it:





Mrs. Jones smiled as she saw them finding mirror images. She had a little activity that she thought they should play. She called the game "REFLECT" because it might help them locate the mirror images of points in Squareville.

Have the class play the game. Here are the instructions for the game.

1. Divide the class into two teams. Let one team choose a highway that is north of y = x. The other team chooses a different highway but it must be south of y = x.

2. Have both teams mark their highway with colored chalk.

3. One team begins by pointing to an intersection on their highway, and then labeling it (e. g. 6, 3).

4. The other team then picks out the mirror image of that intersection on and writes the address at that intersection. The referee merely says, "correct" or "not correct" without giving an explanation.

5. If the mirror image is chosen, the second team gets a point. If it isn't correct, the first team gets a point.

6. Then team two picks a point on its highway and team one finds the mirror image.

7. The first team to get five points wins the game. The teams alternate starting positions.

Referee's Clues

a. Choose point (3, 2). The mirror image point is (2, 3). Just reverse the position of the members of the ordered pair.

b. Another technique is to mark down the highway for each group on Squareville.

Fold the map along $y = x$. As a point is marked, poke a hole through the paper at this point. The hole on the other half of the paper will locate the mirror image.